October 2, 2025 Séminaire ANEDP – Université de Lille

A-posteriori-steered h- and p-robust multigrid and optimal complexity of AFEM

Ani Miraçi
joint work with Philipp Bringmann, Michael Innerberger, Jan Papež, Dirk Praetorius, Julian Streitberger, Martin Vohralík

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Laboratoire Jacques-Louis Lions

Motivation and context

Workflow

MODELING

PDE

$u^\star \in \mathcal{X}$

estimator η_ℓ $\approx$ discretization error $\|\nabla(u^\star - u_\ell^\star)\|$ $\longrightarrow$

DISCRETIZATION(ℓ)

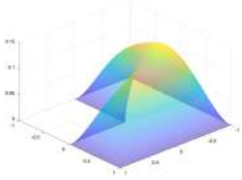
FEM (mesh $\mathcal{T}_\ell$, polynomial degree p)

$u_\ell^\star \in \mathcal{X}_\ell^p$

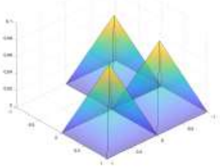
estimator ζ_ℓ^k $\approx$ solver error $\|\nabla(u_\ell^\star - u_\ell^k)\|$ $\longrightarrow$

SOLVER(k)

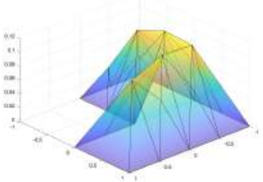
$u_\ell^k \in \mathcal{X}_\ell^p$



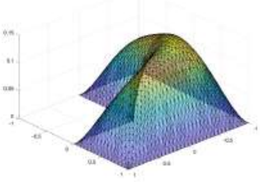
solution $u^\star$



$\ell = 0$
$k = 1$



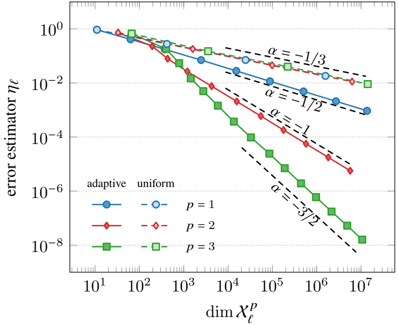
$\ell = 1$
$k = 3$



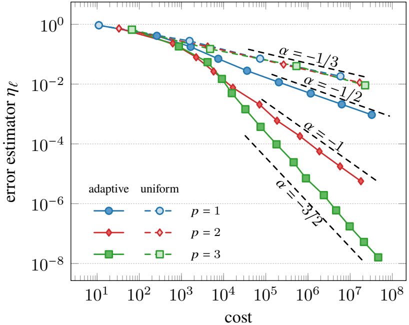
$\ell = 8$
$k = 0$

1. **Reliability** $\|\nabla(u^\star - u_\ell^k)\| \lesssim \eta_\ell + \zeta_\ell^k$
2. **Convergence** $u_\ell^k \longrightarrow u^\star$
3. **Optimal** rates of convergence

Optimal convergence $\alpha = p/d$ wrt dofs and cost



$$\sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^{\alpha} \eta_\ell < \infty$$



$$\sup_{\ell \in \mathbb{N}_0} \Big(\sum_{\ell'=0}^{\ell} \sum_{k'=0}^{k[\ell']} \#\mathcal{T}_{\ell'} \Big)^{\alpha} \eta_\ell < \infty$$

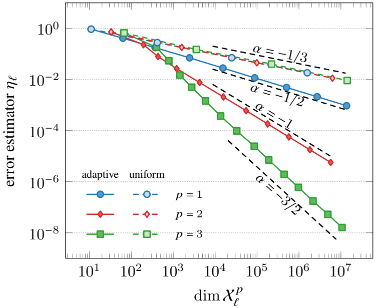
Binev, Dahmen, DeVore: *Numer. Math.*, 97 (2004)

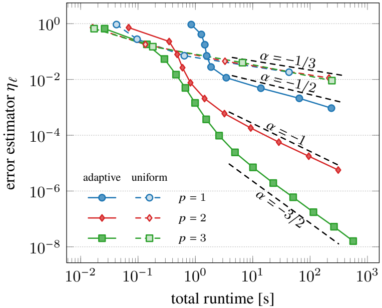
Stevenson: *Found. Comput. Math.*, 7 (2007)

Cascón, Kreuzer, Nochetto, Siebert: *SIAM J. Numer. Anal.*, 46 (2008)

Gantner, Haberl, Praetorius, Schimanko: *Math. Comp.*, 90 (2021)

Towards **optimal complexity**





Optimal complexity of AFEM requires each of its modules to be realized in linear complexity:

- SOLVE is critical
- ESTIMATE ✓
- MARK (Stevenson 2007, Pfeiler-Praetorius 2020 for minimal cardinality marking) ✓
- REFINE (Binev-Dahmen-DeVore 2004, Stevenson 2008) ✓

Optimal algebraic solver

Model problem and discretization

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Consider the **symmetric linear elliptic PDE**

$$-\mathrm{div}(\boldsymbol{A}\nabla u^\star) = f \quad \text{in } \Omega \subset \mathbb{R}^d \qquad u^\star = 0 \quad \text{on } \partial\Omega$$

with **weak formulation** searching for $u^\star \in \mathcal{X} := H_0^1(\Omega)$ solution of

$$\langle \boldsymbol{A}\nabla u^\star \,,\, \nabla v\rangle_{L^2(\Omega)} = \langle f \,,\, v\rangle_{L^2(\Omega)} \quad \forall v \in \mathcal{X}$$

Lax-Milgram framework $\Longrightarrow$ *well-posed problem*

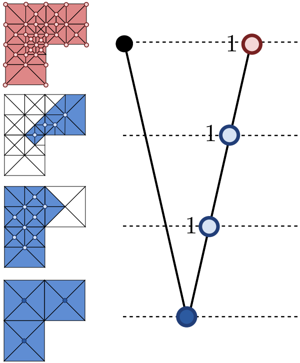
FEM discretization

- $\mathcal{T}_\ell$ simplicial mesh of Ω and $p \geq 1$ polynomial degree
- finite element space $\mathcal{X}_\ell^p := \{v_\ell \in H_0^1(\Omega) : v_\ell|_T \in \mathbb{P}_p(T)\ \forall T \in \mathcal{T}_\ell\} \subset \mathcal{X}$
- search for $u_\ell^\star \in \mathcal{X}_\ell^p$ solution of

$$\langle \boldsymbol{A}\nabla u_\ell^\star \,,\, \nabla v_\ell\rangle_{L^2(\Omega)} = \langle f \,,\, v_\ell\rangle_{L^2(\Omega)} \quad \forall v_\ell \in \mathcal{X}_\ell^p$$

A-posteriori-steered multigrid

$u_\ell^k \approx u_\ell^\star \qquad u_\ell^{k+1} = u_\ell^k + \lambda_0 \rho_0 + \sum_{\ell'=1}^{\ell-1} \lambda_{\ell'} \sum_{z \in \mathcal{V}_{\ell'}^+} \rho_{\ell',z} + \lambda_\ell \sum_{z \in \mathcal{V}_\ell} \rho_{\ell,z}$



V-cycle :

- no pre- and one post-smoothing step
- lowest-order coarse solve
- lowest-order and local smoothing in intermediate levels : h-**robustness**
  $\Longrightarrow$ *Jacobi iteration*
- high-order and patch-wise smoothing on finest level: p-**robustness**
  $\Longrightarrow$ *additive Schwarz* / *block Jacobi* with overlap
- level-wise optimal steps in error correction
  $\Longrightarrow$ *line search* : $\operatorname*{argmin}_{\lambda \in \mathbb{R}} \|\nabla(u_\ell^\star - (u + \lambda\rho))\|$
- built-in a posteriori estimator
  $\Longrightarrow$ $\zeta_\ell^k = \lambda_0 \|\nabla \rho_0\|^2 + \sum_{\ell'=1}^{\ell-1} \lambda_{\ell'} \sum_{z \in \mathcal{V}_{\ell'}^+} \|\nabla \rho_{\ell',z}\|^2 + \lambda_\ell \sum_{z \in \mathcal{V}_\ell} \|\nabla \rho_{\ell,z}\|^2$

Chen, Nochetto, Xu: *Numer. Math.*, 120 (2012)

Schöberl, Melenk, Pechstein, Zaglmayr: *IMA J. Numer. Anal.*, 28 (2008)

Heinrichs: *J. Comput. Phys.*, 77 (1988)

Solver contraction

Theorem (h- and p-robust contraction of the algebraic error)

$$\|\nabla(u_\ell^\star - u_\ell^{k+1})\| \le q_{\rm alg}\,\|\nabla(u_\ell^\star - u_\ell^k)\| \quad 0 < q_{\rm alg}(d, \boldsymbol{A}, \mathcal{T}_0) < 1$$

Theorem (h- and p-robust reliability and efficiency of the a posteriori estimator)

$$\zeta_\ell^k \le \|\nabla(u_\ell^\star - u_\ell^k)\| \quad \text{and} \quad \|\nabla(u_\ell^\star - u_\ell^k)\| \le C_{\rm rel}\,\zeta_\ell^k \quad C_{\rm rel} := [2/(1-q_{\rm alg}^2)]^{1/2}$$

solver contraction $\iff$ *reliability of the a posteriori estimator*

Corollary (equivalence algebraic error – localized a posteriori estimator)

$$\|\nabla(u_\ell^\star - u_\ell^k)\|^2 \approx (\zeta_\ell^k)^2 = \|\nabla\rho_0\|^2 + \sum_{\ell'=1}^{\ell-1} \lambda_{\ell'} \sum_{z\in\mathcal{V}_{\ell'}^+} \|\nabla\rho_{\ell',z}\|^2 + \lambda_\ell \sum_{z\in\mathcal{V}_\ell} \|\nabla\rho_{\ell,z}\|^2$$

Miraçi, Papež, Vohralík: *SIAM J. Sci. Comput.*, 43 (2021)

Innerberger, Miraçi, Praetorius, Streitberger: *ESAIM Math. Model. Numer. Anal.*, 58 (2024)

Adaptive mesh refinement and algebraic solver

AFEM with iterative solver

input: initial mesh $\mathcal{T}_0$, adaptivity parameters $0 < \theta \leq 1$, $\lambda_{\mathrm{alg}} > 0$

for each $\ell = 0, 1, 2, \ldots$ **repeat** (mesh-refinement loop)

SOLVE & ESTIMATE (algebraic solver loop)

for $k = 1, 2, \ldots, K$, **repeat**

compute the new approximation $u_\ell^k \approx u_\ell^\star$

compute the estimators of algebraic error ζ_ℓ^k and of discretization error $\eta_\ell(u_\ell^k)$

until $\zeta_\ell^k \leq \lambda_{\mathrm{alg}} \eta_\ell(u_\ell^k)$

MARK select $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^K)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^K)^2$

REFINE $\mathcal{T}_{\ell+1} := \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

$u_{\ell+1}^0 := u_\ell^K$

output: approximations u_ℓ^K and corresponding estimators $\zeta_\ell^K, \eta_\ell(u_\ell^K)$

Reliability and linear convergence

A posteriori control for each approximation u_ℓ^k

$$\underbrace{\|\nabla(u^\star - u_\ell^k)\|}_{\text{total error}} \leq \underbrace{\|\nabla(u^\star - u_\ell^\star)\|}_{\text{discretization error}} + \underbrace{\|\nabla(u_\ell^\star - u_\ell^k)\|}_{\text{algebraic error}} \overset{\text{reliability (discretization)}}{\lesssim} \underbrace{\eta_\ell(u_\ell^\star)}_{\text{discretization estimator}} + \|\nabla(u_\ell^\star - u_\ell^k)\| =: \underbrace{\mathrm{H}_\ell^k}_{\text{quasi-error}}$$

$$\overset{\text{reliability (algebra)}}{\lesssim} \eta_\ell(u_\ell^k) + \underbrace{\zeta_\ell^k}_{\text{algebra estimator}}$$

Theorem (full R-linear convergence of the quasi-error) Consider arbitrary $0 < \theta \leq 1$, $\lambda_{\mathrm{alg}} > 0$. Then

$$\mathrm{H}_\ell^k \leq C_{\mathrm{lin}}\, q_{\mathrm{lin}}^{|\ell,k|-|\ell',k'|}\, \mathrm{H}_{\ell'}^{k'}, \quad C_{\mathrm{lin}} > 0, \; 0 < q_{\mathrm{lin}} < 1$$

Contraction independantly of the algorithmic step: mesh-refinement or algebraic iteraiton.

Corollary (convergence of the total error)

$$\|\nabla(u^\star - u_\ell^k)\| \lesssim \mathrm{H}_\ell^k \lesssim q_{\mathrm{lin}}^{|\ell,k|}\, \mathrm{H}_0^0 \longrightarrow 0 \quad \text{for} \quad |\ell,k| \longrightarrow \infty$$

Full R-linear convergence implies rates = complexity

- $\mathfrak{R}(\alpha) := \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^\alpha \mathrm{H}_\ell^k < \infty$ rate α wrt dofs is possible
- $\widehat{\mathfrak{R}}(\alpha) := \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\leq|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^\alpha \mathrm{H}_\ell^k < \infty$ rate α wrt costs/overall computational cost is possible

Proposition

$$\mathfrak{R}(\alpha) \leq \widehat{\mathfrak{R}}(\alpha) \leq \frac{C_{\mathrm{lin}}}{(1-q_{\mathrm{lin}}^{1/\alpha})^\alpha} \mathfrak{R}(\alpha)$$

- Proof: $\#\mathcal{T}_{\ell'} \leq \mathfrak{R}(\alpha)^{\frac{1}{\alpha}} (\mathrm{H}_{\ell'}^{k'})^{-\frac{1}{\alpha}} \quad \forall (\ell',k') \in \mathcal{Q}$, summing and using the geometric series :

$$\Longrightarrow \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\leq|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^\alpha \leq \mathfrak{R}(\alpha)^{\frac{1}{\alpha}} \sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\leq|\ell,k|}} (\mathrm{H}_{\ell'}^{k'})^{-\frac{1}{\alpha}} \leq \mathfrak{R}(\alpha)^{\frac{1}{\alpha}} C_{\mathrm{lin}}^{\frac{1}{\alpha}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\leq|\ell,k|}} q_{\mathrm{lin}}{}^{(|\ell,k|-|\ell',k'|)/\alpha} \Big) (\mathrm{H}_\ell^k)^{-\frac{1}{\alpha}}$$

Carstensen, Feischl, Page, Praetorius: *Comput. Math. Appl.*, 67 (2014)

Optimal complexity

We introduce the notion of *approximation class* : $\|u^\star\|_{\mathbb{A}_\alpha} := \sup_{N \geq \#\mathcal{T}_0} N^\alpha [\min_{\#\mathcal{T}_{\mathrm{opt}} \leq N} \eta_{\mathrm{opt}}]$

$$\|u^\star\|_{\mathbb{A}_\alpha} < \infty \quad \Longleftrightarrow \quad u^\star \text{ can be approximated with rate } \alpha \text{ wrt dofs}$$

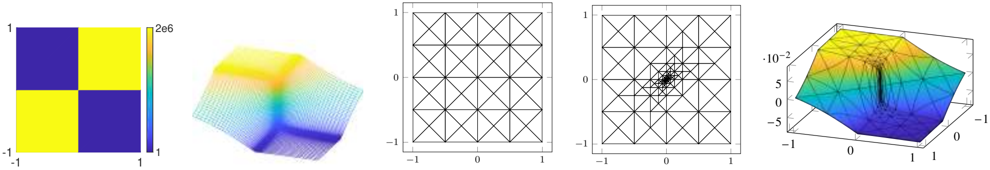
Theorem (optimal convergence wrt to overall computational cost)
Let $\alpha > 0$ such that $\|u^\star\|_{\mathbb{A}_\alpha} < \infty$. Suppose $0 < \theta < 1$ et $\lambda_{\mathrm{alg}} > 0$ sufficiently small

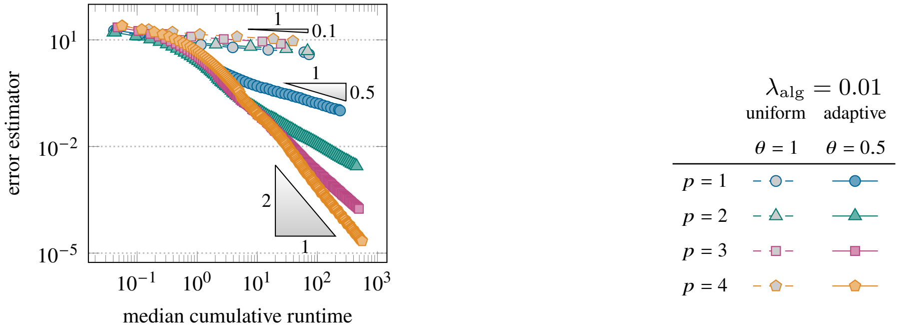
$$\Longrightarrow \quad \|u^\star\|_{\mathbb{A}_\alpha} \lesssim \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\leq|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^\alpha \mathrm{H}_\ell^k \lesssim \max\big\{\|u^\star\|_{\mathbb{A}_\alpha}, \mathrm{H}_0^0)\big\}$$

if $u^\star$ *can be approximated with rate* α *wrt dofs*
then AFEM with algebraic solver approximates $u^\star$ *with the rate* α *wrt cost.*

Gantner, Haberl, Praetorius, Schimanko: *Math. Comp.*, 90 (2021)
Innerberger, Miraçi, Praetorius, Streitberger: *ESAIM Math. Model. Numer. Anal.*, 58 (2024)

Optimal complexity of AFEM





Kellogg: *Appl. Anal.*, 4 (1975)

Optimality and robust contraction of the solver

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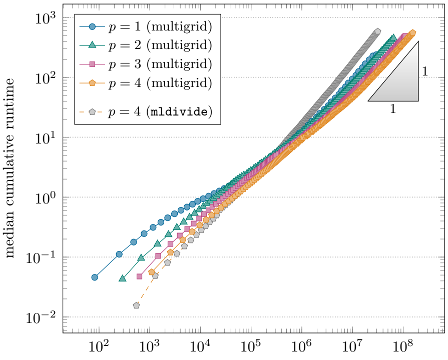
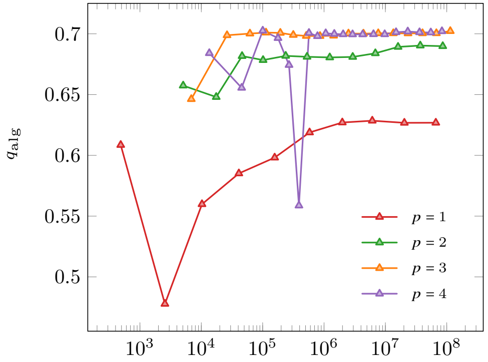
L-shaped domain: $\quad -\Delta u^\star = 1 \quad$ on $\quad \Omega, \quad u^\star = 0 \quad$ in $\quad \partial\Omega$



overall computational cost

$\theta = 0.5, \ \lambda_{\mathrm{alg}} = 0.01$

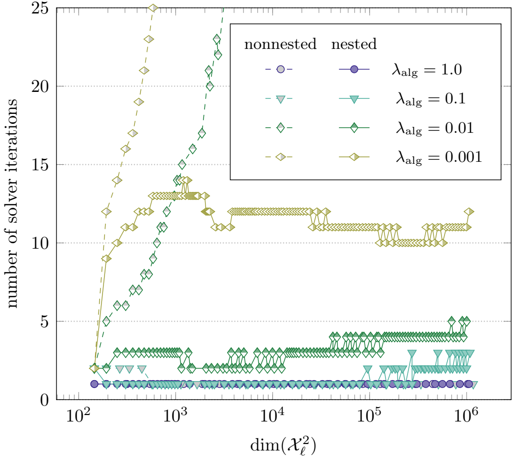


overall computational cost

$\theta = 0.5, \ \lambda_{\mathrm{alg}} = 10^{-5}$

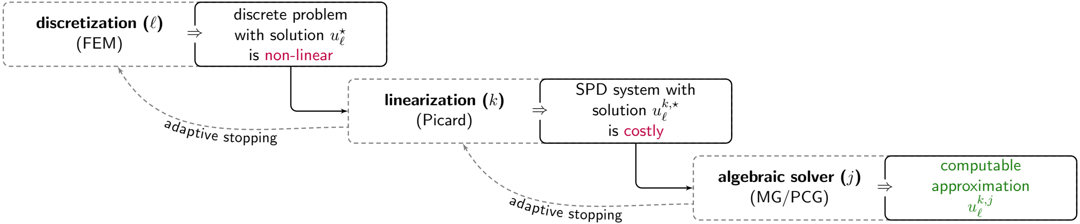
Importance of nested iterations

L-shaped domain: $-\Delta u^\star = 1$ on Ω, $u^\star = 0$ in $\partial\Omega$



Adaptivity with non-linear solvers

Adaptive approach with nested solvers



$\Longrightarrow$ three nested loops: mesh-refinement (ℓ) $\mapsto$ linearization (k) $\mapsto$ algebra (j)

$\Longrightarrow$ each of the solvers (algebra and linearization) is contractive

Congreve, Wihler: *J. Comput. Appl. Math.*, 311 (2017)

Heid, Praetorius, Wihler: *Comput. Methods Appl. Math.*, 21 (2021)

Haberl, Praetorius, Schimanko, Vohralík: *Numer. Math.*, 147 (2021)

Non-linear problem with energy

Scalar non-linearity

$M\,(t-s) \leq \mu(t^2)t - \mu(s^2)s \leq L\,(t-s) \quad \forall\, 0 \leq s \leq t$

$$-\mathrm{div}\,\big(\mu(|\nabla u^\star|^2)\nabla u^\star\big) = f \quad \text{in } \Omega, \quad u^\star = 0 \quad \text{on } \partial\Omega$$

Weak formulation find $u^\star \in H^1_0(\Omega)$ such that

$$\langle \mathcal{A}\, u^\star\,,\, v\rangle := \langle \mu(|\nabla u^\star|^2)\nabla u^\star\,,\, \nabla v\rangle_{L^2(\Omega)} = F(v) \quad \forall\, v \in H^1_0(\Omega)$$

- strongly monotone $\quad M\,\|\nabla(u-v)\|^2 \leq \langle \mathcal{A}u - \mathcal{A}v\,,\, u-v\rangle$
- Lipschitz continuous $\quad \langle \mathcal{A}u - \mathcal{A}v\,,\, w\rangle \leq L\,\|\nabla(u-v)\|\,\|\nabla w\|$

$\Longrightarrow$ existence and uniqueness of the weak solution $u^\star \in H^1_0(\Omega)$

Energy framework

- Energy $\quad \mathcal{E}(v) := \frac{1}{2}\int_\Omega \int_0^{|\nabla v(x)|^2} \mu(t)\,\mathrm{d}t\,\mathrm{d}x - F(v)$
- Distance $\quad \mathbb{D}^2(u,v) := \mathcal{E}(v) - \mathcal{E}(u) \quad$ remark: $\mathbb{D}^2(u^\star, v) \approx \|\nabla(u^\star - v)\|^2$

Zeidler: (1990)

Adaptive FEM with nested iterative solvers

input: initial mesh $\mathcal{T}_0$, initial guess $u_0^{0,0}$, adaptivity parameters $0 < \theta \le 1$, $\lambda_{\rm lin} > 0$

for each $\ell = 0, 1, 2, \dots$ **repeat** (mesh-refinement loop)

SOLVE & ESTIMATE (linearization loop)

for $k = 1, 2, \dots, K$, **repeat**

for $j = 1, 2, \dots, J$, **repeat** (algebra loop)

compute $u_\ell^{k,j} \approx u_\ell^{k,\star}$ from the previous step $u_\ell^{k,j-1}$

compute the local indicators $\eta_\ell(T, u_\ell^k)$ for all $T \in \mathcal{T}_\ell$

until [algebra-criterion]

until $\mathbb{D}^2(u_\ell^{k,J}, u_\ell^{k-1,J}) \le \lambda_{\rm lin}\, \eta_\ell(u_\ell^{k,J})^2$

MARK choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^{K,J})^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^{K,J})^2$

REFINE $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

$u_{\ell+1}^{0,0} := u_\ell^{K,J}$

output: discrete solutions $u_\ell^{K,J}$ and corresponding estimators $\eta_\ell(u_\ell^{K,J})$

Termination of algebraic solver

- the stopping criterion should guarantee that nested linearization-algebraic solver contracts in energy

$$\mathbb{D}^2(u_\ell^\star, u_\ell^{k,J}) \le q_{\mathrm{ctr}}\, \mathbb{D}^2(u_\ell^\star, u_\ell^{k-1,J}) \qquad 0 < q_{\mathrm{ctr}} < 1$$

Equilibration criterion [HPSV21]

- stop algebraic solver if $\|\nabla(u_\ell^{k,J} - u_\ell^{k,J-1})\|^2 \le \lambda_{\mathrm{alg}} \left[\eta_\ell(u_\ell^{k,J})^2 + \mathbb{D}^2(u_\ell^{k,J}, u_\ell^{k,J-1}) \right]$
- stop linearization if $\mathbb{D}^2(u_\ell^{K,J}, u_\ell^{K-1,J}) \le \lambda_{\mathrm{lin}}\, \eta_\ell(u_\ell^{K,J})$

$\Longrightarrow$ full R-linear convergence for arbitrary λ_{lin} but sufficiently small λ_{alg}

Note: there exists $C_{\mathrm{nrg}}^\star > 0$ st. $C_{\mathrm{nrg}}^\star \|\nabla(u_\ell^{k,\star} - u_\ell^{k-1,J})\|^2 \le \mathbb{D}^2(u_\ell^{k,\star}, u_\ell^{k-1,J})$

Energy-based criterion [MPS24+]

- enforce algorithmically $\|\nabla(u_\ell^{k,J} - u_\ell^{k,J-1})\|^2 \lesssim \mathbb{D}^2(u_\ell^{k,J}, u_\ell^{k,J-1})$ **(parameter-free)**
- stop linearization if $\mathbb{D}^2(u_\ell^{K,J}, u_\ell^{K-1,J}) \le \lambda_{\mathrm{lin}}\, \eta_\ell(u_\ell^{K,J})$

$\Longrightarrow$ full R-linear convergence for arbitrary $\lambda_{\mathrm{lin}} > 0$

Haberl, Praetorius, Schimanko, Vohralík: *Numer. Math.*, 147 (2021)

Miraçi, Praetorius, Streitberger: *Math. Comp. accepted* (2025)

Optimality of AFEM with linearization : $p \geq 1$

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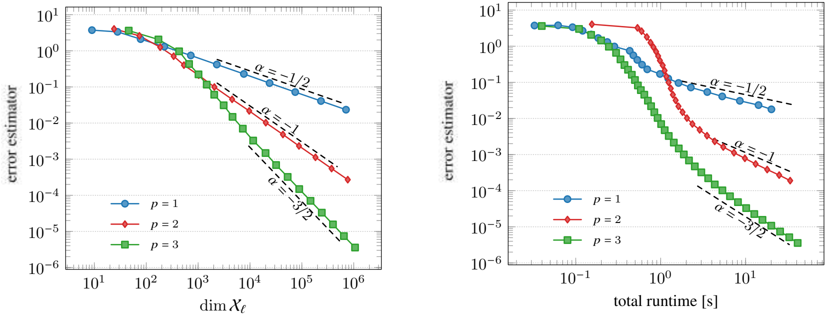
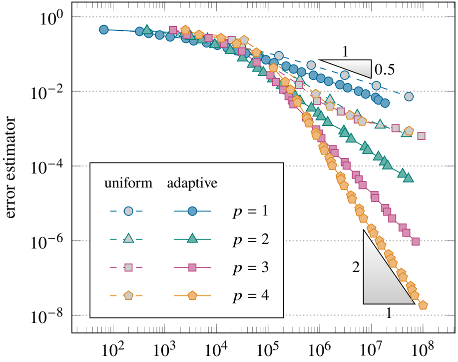
$$-\operatorname{div}\big(\nabla u^\star + \exp(-|\nabla u^\star|^2)\nabla u^\star\big) = 1 \quad \text{on} \quad \Omega, \qquad u^\star = 0 \quad \text{in} \quad \partial\Omega$$



Miraçi, Praetorius, Streitberger: *Math. Comp. accepted* (2025)

Optimality of AFEM with symmetrization : $p \geq 1$

$$-\Delta u^\star(x) + x \cdot \nabla u^\star(x) + u^\star(x) = 1 \quad \text{in} \quad \Omega, \qquad u^\star(x) = 0 \quad \text{on} \quad \partial\Omega$$



overall computational cost

Brunner, Innerberger, Miraçi, Praetorius, Streitberger, Heid: *IMA J. Numer. Anal.*, 44 (2024)

Take home messages

1. analysis of AFEM should focus rather on rates wrt. *complexity/time* than dofs
2. *linear complexity* and *contraction* of the iterative solver is crucial
3. use *nested* iterations and *termination* of solver balancing the different error components
4. *reliability* via a posteriori error estimators is ensured
5. full R-linear convergence
   - gives *contraction* regardless of algorithmic step
   - holds for *arbitrary* adaptivity parameters
   - provides the equivalence *rates* = *complexity*
6. *optimal complexity* is ensured for sufficiently small parameters
7. every *achievable* convergence rate is produced by AFEM
8. setting of certain non-symmetric and non-linear problems fits within this adaptive framework

Selected contributions

Miraçi, Papež, Vohralík

A-posteriori-steered p-robust multigrid with optimal step-sizes and adaptive number of smoothing steps

SIAM J. Sci. Comput., 43, DOI: `10.1137/20M1349503` (2021)

Innerberger, Miraçi, Praetorius, Streitberger

hp-robust multigrid solver on locally refined meshes for FEM discretizations of symmetric elliptic PDEs

ESAIM Math. Model. Numer. Anal., 58, DOI: `10.1051/m2an/2023104` (2024)

Miraçi, Praetorius, Streitberger

Unconditional full linear convergence and optimal complexity of adaptive iteratively linearized FEM for nonlinear PDEs

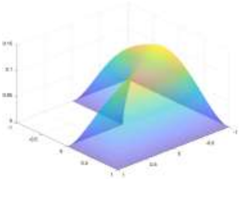
Math. Comp. accepted (2025)

Bringmann, Miraçi, Praetorius

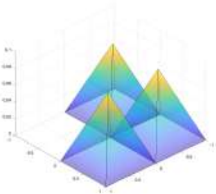
Chapter Four - Iterative solvers in adaptive FEM: Adaptivity yields quasi-optimal computational runtime

Advances in Applied Mechanics, Elsevier, 59, DOI: `https://doi.org/10.1016/bs.aams.2024.08.002` (2024)

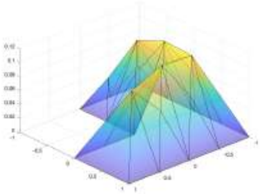
Thank you for your attention!



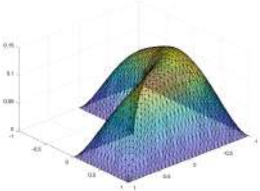
solution $u^\star$



$\ell = 0$
$k = 1$



$\ell = 1$
$k = 3$



$\ell = 8$
$k = 0$