

October 10, 2025 Séminaire du Laboratoire Jacques-Louis Lions

A-posteriori-steered h - and p -robust multigrid and optimal complexity of AFEM

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joint work with Philipp Bringmann, Michael Innerberger, Jan Papež, Dirk Praetorius, Julian Streitberger, Martin Vohralík



[slides](#)



Laboratoire Jacques-Louis Lions

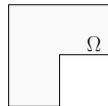
What is it all about?

Example

Poisson equation: $-\Delta u^* = 1$ on Ω , $u^* = 0$ in $\partial\Omega$ (L-shaped domain)

\Rightarrow Find $u^* \in H_0^1(\Omega)$ solution of

$$\langle \nabla u^*, \nabla v \rangle_{L^2(\Omega)} = \langle 1, v \rangle_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$$

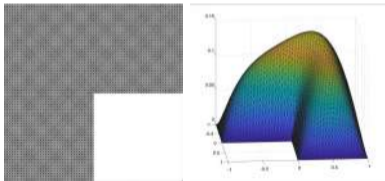


Finite Element Method: choose mesh \mathcal{T}_ℓ and polynomial degree $p \geq 1$

\Rightarrow Find $u_\ell^* \in \mathcal{X}_\ell^p := \{v_\ell \in H_0^1(\Omega) : v_\ell|_T \in \mathbb{P}_p(T) \quad \forall T \in \mathcal{T}_\ell\}$ solution of

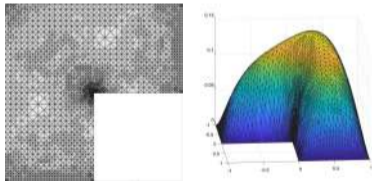
$$\langle \nabla u_\ell^*, \nabla v_\ell \rangle_{L^2(\Omega)} = \langle 1, v_\ell \rangle_{L^2(\Omega)} \quad \forall v_\ell \in \mathcal{X}_\ell^p$$

Option 1: Uniform mesh



dofs: 24833 error: 2.32e-02

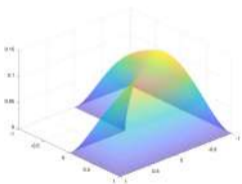
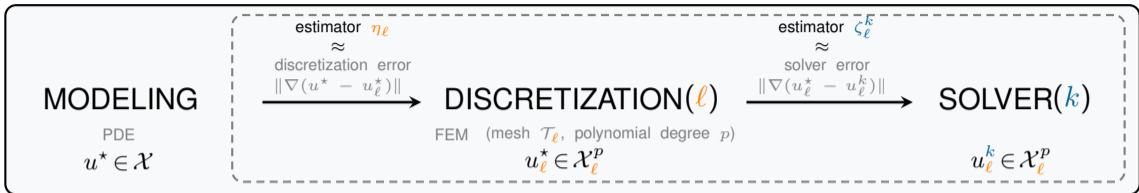
Option 2: Graded mesh



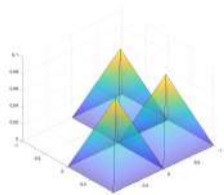
dofs: 13197 error: 1.75e-03

Question: How to obtain such a choice of mesh?

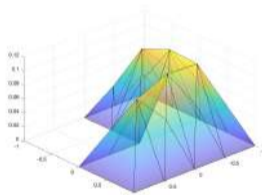
\Rightarrow AFEM: SOLVE – ESTIMATE – MARK – REFINE



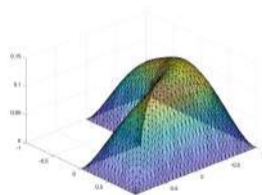
solution u^*



$\ell = 0$
 $k = 1$



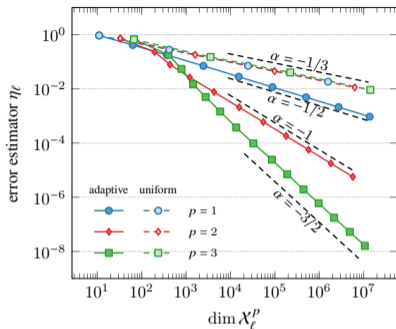
$\ell = 1$
 $k = 3$



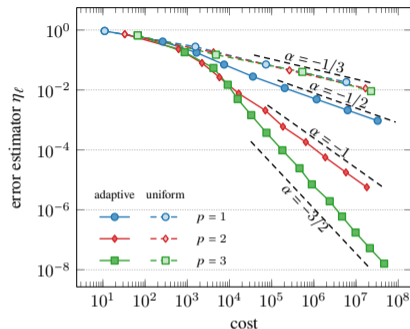
$\ell = 8$
 $k = 0$

1. **Reliability** $\|\nabla(u^* - u_\ell^k)\| \lesssim \eta_\ell + \zeta_\ell^k$
2. **Convergence** $u_\ell^k \rightarrow u^*$
3. **Optimal** rates of convergence





Optimal convergence $\alpha = p/d$ wrt **dofs** and **cost**



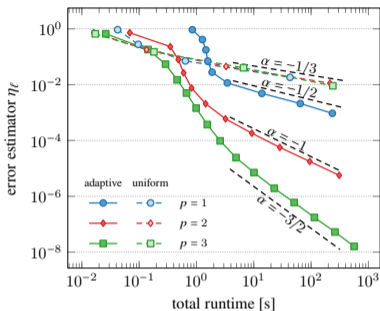
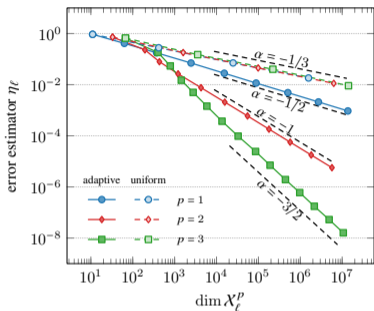
$$\sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^\alpha \eta_\ell < \infty$$



$$\sup_{\ell \in \mathbb{N}_0} \left(\sum_{\ell'=0}^{\ell} \sum_{k'=0}^{k[\ell']} \#\mathcal{T}_{\ell'} \right)^\alpha \eta_\ell < \infty$$

-  Binev, Dahmen, DeVore: *Numer. Math.*, 97 (2004)
-  Stevenson: *Found. Comput. Math.*, 7 (2007)
-  Cascón, Kreuzer, Nochetto, Siebert: *SIAM J. Numer. Anal.*, 46 (2008)
-  Gantner, Haberl, Praetorius, Schimanko: *Math. Comp.*, 90 (2021)

Towards optimal complexity



Optimal complexity of AFEM requires each of its modules to be realized in linear complexity:

- SOLVE is **critical**
- ESTIMATE ✓
- MARK (Stevenson 2007, Pfeiler-Praetorius 2020 for minimal cardinality marking) ✓
- REFINE (Binev-Dahmen-DeVore 2004, Stevenson 2008) ✓

Optimal algebraic solver

Model problem and discretization

Consider the **symmetric linear elliptic PDE**

$$-\operatorname{div}(\mathbf{A}\nabla u^*) = f \quad \text{in } \Omega \subset \mathbb{R}^d \quad u^* = 0 \quad \text{on } \partial\Omega$$

with **weak formulation** searching for $u^* \in \mathcal{X} := H_0^1(\Omega)$ solution of

$$\langle \mathbf{A}\nabla u^*, \nabla v \rangle_{L^2(\Omega)} = \langle f, v \rangle_{L^2(\Omega)} \quad \forall v \in \mathcal{X}$$

Lax-Milgram framework \implies well-posed problem

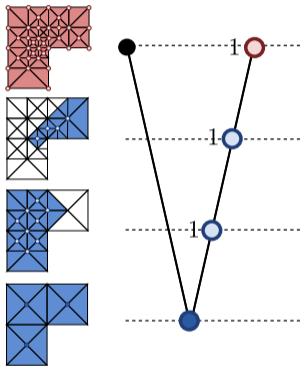
FEM discretization

- \mathcal{T}_ℓ simplicial mesh of Ω and $p \geq 1$ polynomial degree
- finite element space $\mathcal{X}_\ell^p := \{v_\ell \in H_0^1(\Omega) : v_\ell|_T \in \mathbb{P}_p(T) \quad \forall T \in \mathcal{T}_\ell\} \subset \mathcal{X}$
- search for $u_\ell^* \in \mathcal{X}_\ell^p$ solution of

$$\langle \mathbf{A}\nabla u_\ell^*, \nabla v_\ell \rangle_{L^2(\Omega)} = \langle f, v_\ell \rangle_{L^2(\Omega)} \quad \forall v_\ell \in \mathcal{X}_\ell^p$$

A-posteriori-steered multigrid

$$u_\ell^k \approx u_\ell^* \quad u_\ell^{k+1} = u_\ell^k + \lambda_0 \rho_0 + \sum_{\ell'=1}^{\ell-1} \lambda_{\ell'} \sum_{z \in \mathcal{V}_{\ell'}^+} \rho_{\ell',z} + \lambda_\ell \sum_{z \in \mathcal{V}_\ell} \rho_{\ell,z}$$



V-cycle :

- no pre- and **one** post-smoothing step
- **lowest-order** coarse solve
- **lowest-order** and **local** smoothing in intermediate levels : ***h-robustness***
 \implies *Jacobi iteration*
- **high-order** and **patch-wise** smoothing on finest level: ***p-robustness***
 \implies *additive Schwarz / block Jacobi with overlap*

- level-wise **optimal** steps in error correction
 \implies *line search* : $\operatorname{argmin}_{\lambda \in \mathbb{R}} \|\nabla(u_\ell^* - (u + \lambda\rho))\|$

- **built-in** a posteriori estimator

$$\implies \zeta_\ell^k = \lambda_0 \|\nabla \rho_0\|^2 + \sum_{\ell'=1}^{\ell-1} \lambda_{\ell'} \sum_{z \in \mathcal{V}_{\ell'}^+} \|\nabla \rho_{\ell',z}\|^2 + \lambda_\ell \sum_{z \in \mathcal{V}_\ell} \|\nabla \rho_{\ell,z}\|^2$$

📄 Chen, Nochetto, Xu: *Numer. Math.*, 120 (2012)

📄 Schöberl, Melenk, Pechstein, Zaglmayr: *IMA J. Numer. Anal.*, 28 (2008)

📄 Heinrichs: *J. Comput. Phys.*, 77 (1988)

Solver contraction

Theorem (*h-* and *p-robust contraction* of the algebraic error)

$$\|\nabla(u_\ell^* - u_\ell^{k+1})\| \leq q_{\text{alg}} \|\nabla(u_\ell^* - u_\ell^k)\| \quad 0 < q_{\text{alg}}(d, \mathbf{A}, \mathcal{T}_0) < 1$$

Theorem (*h-* and *p-robust reliability* and *efficiency* of the a posteriori estimator)

$$\zeta_\ell^k \leq \|\nabla(u_\ell^* - u_\ell^k)\| \quad \text{and} \quad \|\nabla(u_\ell^* - u_\ell^k)\| \leq C_{\text{rel}} \zeta_\ell^k \quad C_{\text{rel}} := [2/(1 - q_{\text{alg}}^2)]^{1/2}$$

solver contraction \iff *reliability of the a posteriori estimator*

Corollary (*equivalence* algebraic error – *localized* a posteriori estimator)

$$\|\nabla(u_\ell^* - u_\ell^k)\|^2 \approx (\zeta_\ell^k)^2 = \|\nabla\rho_0\|^2 + \sum_{\ell'=1}^{\ell-1} \lambda_{\ell'} \sum_{z \in \mathcal{V}_{\ell'}^+} \|\nabla\rho_{\ell',z}\|^2 + \lambda_\ell \sum_{z \in \mathcal{V}_\ell} \|\nabla\rho_{\ell,z}\|^2$$

 Miraçi, Papež, Vohralík: *SIAM J. Sci. Comput.*, 43 (2021)

 Innerberger, Miraçi, Praetorius, Streitberger: *ESAIM Math. Model. Numer. Anal.*, 58 (2024)

Adaptive mesh refinement and algebraic solver

AFEM with iterative solver

input: initial mesh \mathcal{T}_0 , adaptivity parameters $0 < \theta \leq 1$, $\lambda_{\text{alg}} > 0$

for each $\ell = 0, 1, 2, \dots$ **repeat**

(mesh-refinement loop)

SOLVE & ESTIMATE

(algebraic solver loop)

for $k = 1, 2, \dots, K$, **repeat**

compute the new approximation $u_\ell^k \approx u_\ell^*$

compute the estimators of algebraic error ζ_ℓ^k and of discretization error $\eta_\ell(u_\ell^k)$

until $\zeta_\ell^k \leq \lambda_{\text{alg}} \eta_\ell(u_\ell^k)$

MARK select $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^K)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^K)^2$

REFINE $\mathcal{T}_{\ell+1} := \text{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

$u_{\ell+1}^0 := u_\ell^K$

output: approximations u_ℓ^K and corresponding estimators $\zeta_\ell^K, \eta_\ell(u_\ell^K)$

Reliability and linear convergence

A posteriori control for each approximation u_ℓ^k

$$\underbrace{\|\nabla(u^* - u_\ell^k)\|}_{\text{total error}} \leq \underbrace{\|\nabla(u^* - u_\ell^*)\|}_{\text{discretization error}} + \underbrace{\|\nabla(u_\ell^* - u_\ell^k)\|}_{\text{algebraic error}} \stackrel{\text{reliability (discretization)}}{\lesssim} \underbrace{\eta_\ell(u_\ell^*)}_{\text{discretization estimator}} + \|\nabla(u_\ell^* - u_\ell^k)\| =: \underbrace{H_\ell^k}_{\text{quasi-error}}$$

$$\stackrel{\text{reliability (algebra)}}{\lesssim} \eta_\ell(u_\ell^k) + \underbrace{\zeta_\ell^k}_{\text{algebra estimator}}$$

Define the index set $\mathcal{Q} := \{(\ell, k) \in \mathbb{N}^2 : u_\ell^k \text{ is computed by algorithm}\}$
 with ordering $|\ell, k| := \#\{(\ell', k') \in \mathcal{Q} : u_{\ell'}^{k'} \text{ computed earlier than } u_\ell^k\}$

Theorem (full R-linear convergence of the quasi-error) Consider arbitrary $0 < \theta \leq 1$, $\lambda_{\text{alg}} > 0$. Then

$$H_\ell^k \leq C_{\text{lin}} q_{\text{lin}}^{|\ell, k| - |\ell', k'|} H_{\ell'}^{k'}, \quad C_{\text{lin}} > 0, \quad 0 < q_{\text{lin}} < 1$$

Contraction independently of the algorithmic step: mesh-refinement or algebraic iteration

Corollary (convergence of the total error)

$$\|\nabla(u^* - u_\ell^k)\| \lesssim H_\ell^k \lesssim q_{\text{lin}}^{|\ell, k|} H_0^0 \longrightarrow 0 \quad \text{for} \quad |\ell, k| \longrightarrow \infty$$

Full R-linear convergence implies **rates = complexity**

- $\mathfrak{R}(\alpha) := \sup_{(\ell, k) \in \mathcal{Q}} (\#\mathcal{T}_\ell)^\alpha H_\ell^k < \infty$ rate α wrt dofs is **possible**
- $\widehat{\mathfrak{R}}(\alpha) := \sup_{(\ell, k) \in \mathcal{Q}} \left(\sum_{\substack{(\ell', k') \in \mathcal{Q} \\ |\ell', k'| \leq |\ell, k|}} \#\mathcal{T}_{\ell'} \right)^\alpha H_\ell^k < \infty$ rate α wrt costs/overall computational cost is **possible**

Proposition

$$\mathfrak{R}(\alpha) \leq \widehat{\mathfrak{R}}(\alpha) \leq \frac{C_{\text{lin}}}{(1 - q_{\text{lin}}^{1/\alpha})^\alpha} \mathfrak{R}(\alpha)$$

► **Proof:** $\#\mathcal{T}_{\ell'} \leq \mathfrak{R}(\alpha)^{\frac{1}{\alpha}} (H_{\ell'}^{k'})^{-\frac{1}{\alpha}} \quad \forall (\ell', k') \in \mathcal{Q}$, summing and using the **geometric series** :

$$\Rightarrow \sum_{\substack{(\ell', k') \in \mathcal{Q} \\ |\ell', k'| \leq |\ell, k|}} \#\mathcal{T}_{\ell'} \leq \mathfrak{R}(\alpha)^{\frac{1}{\alpha}} \sum_{\substack{(\ell', k') \in \mathcal{Q} \\ |\ell', k'| \leq |\ell, k|}} (H_{\ell'}^{k'})^{-\frac{1}{\alpha}} \leq \mathfrak{R}(\alpha)^{\frac{1}{\alpha}} C_{\text{lin}}^{\frac{1}{\alpha}} \left(\sum_{\substack{(\ell', k') \in \mathcal{Q} \\ |\ell', k'| \leq |\ell, k|}} q_{\text{lin}}^{(|\ell, k| - |\ell', k'|)/\alpha} \right) (H_\ell^k)^{-\frac{1}{\alpha}}$$

Optimal complexity

We introduce the notion of *approximation class* :

$$\|u^*\|_{\mathbb{A}_\alpha} := \sup_{N \geq \#\mathcal{T}_0} N^\alpha \left[\min_{\#\mathcal{T}_{\text{opt}} \leq N} \eta_{\text{opt}} \right]$$

$$\|u^*\|_{\mathbb{A}_\alpha} < \infty \iff u^* \text{ can be approximated with rate } \alpha \text{ wrt dofs}$$

Theorem (optimal convergence wrt to overall computational cost)

Let $\alpha > 0$ such that $\|u^*\|_{\mathbb{A}_\alpha} < \infty$. Suppose $0 < \theta < 1$ et $\lambda_{\text{alg}} > 0$ **sufficiently small**

$$\implies \|u^*\|_{\mathbb{A}_\alpha} \lesssim \sup_{(\ell, k) \in \mathcal{Q}} \left(\sum_{\substack{(\ell', k') \in \mathcal{Q} \\ |\ell', k'| \leq |\ell, k|}} \#\mathcal{T}_{\ell'} \right)^\alpha H_\ell^k \lesssim \max \{ \|u^*\|_{\mathbb{A}_\alpha}, H_0^0 \}$$

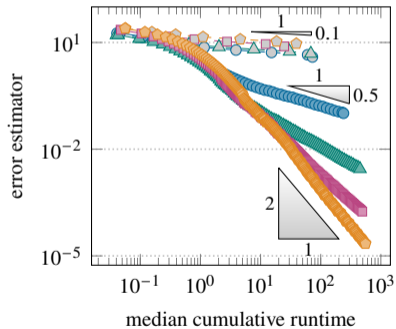
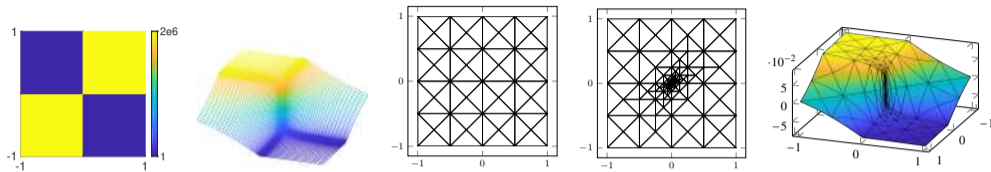
if u^* can be approximated with rate α wrt dofs

then AFEM with algebraic solver approximates u^* with the rate α wrt cost.

 Gantner, Haberl, Praetorius, Schimanko: *Math. Comp.*, 90 (2021)

 Innerberger, Miraçi, Praetorius, Streitberger: *ESAIM Math. Model. Numer. Anal.*, 58 (2024)

Optimal complexity of AFEM

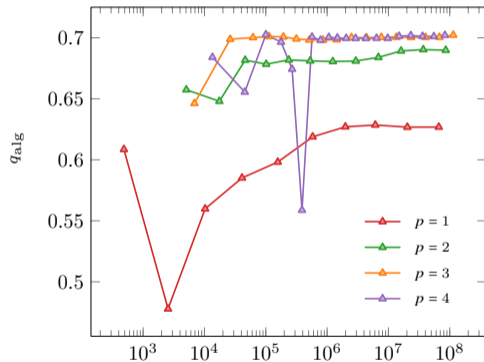
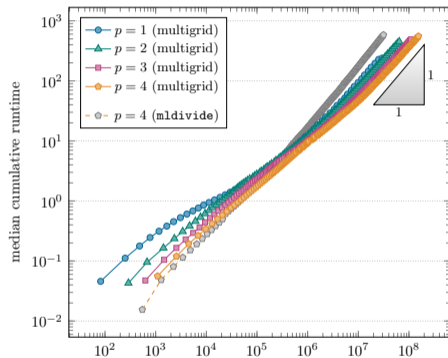


$\lambda_{\text{alg}} = 0.01$

	uniform	adaptive
	$\theta = 1$	$\theta = 0.5$
$p = 1$		
$p = 2$		
$p = 3$		
$p = 4$		

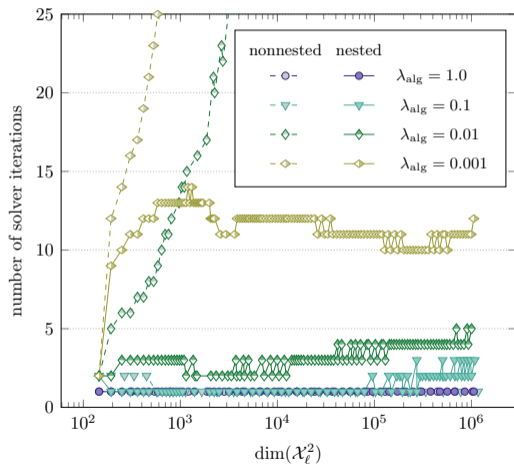
Optimality and robust contraction of the solver

L-shaped domain: $-\Delta u^* = 1$ on Ω , $u^* = 0$ in $\partial\Omega$



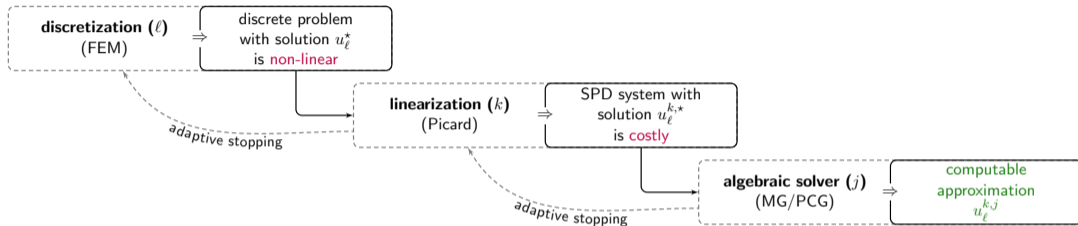
Importance of nested iterations

L-shaped domain: $-\Delta u^* = 1$ on Ω , $u^* = 0$ in $\partial\Omega$



Adaptivity with non-linear solvers

Adaptive approach with nested solvers



⇒ three **nested** loops: mesh-refinement (ℓ) \mapsto linearization (k) \mapsto algebra (j)

⇒ each of the solvers (algebra and linearization) is **contractive**

 Congreve, Wihler: *J. Comput. Appl. Math.*, 311 (2017)

 Heid, Praetorius, Wihler: *Comput. Methods Appl. Math.*, 21 (2021)

 Haberl, Praetorius, Schimanko, Vohralík: *Numer. Math.*, 147 (2021)

Non-linear problem with energy

Scalar non-linearity

$$M(t-s) \leq \mu(t^2)t - \mu(s^2)s \leq L(t-s) \quad \forall 0 \leq s \leq t$$

$$-\operatorname{div}(\mu(|\nabla u^*|^2)\nabla u^*) = f \quad \text{in } \Omega, \quad u^* = 0 \quad \text{on } \partial\Omega$$

Weak formulation find $u^* \in H_0^1(\Omega)$ such that

$$\langle \mathcal{A}u^*, v \rangle := \langle \mu(|\nabla u^*|^2)\nabla u^*, \nabla v \rangle_{L^2(\Omega)} = F(v) \quad \forall v \in H_0^1(\Omega)$$

- strongly monotone $M \|\nabla(u-v)\|^2 \leq \langle \mathcal{A}u - \mathcal{A}v, u - v \rangle$

- Lipschitz continuous $\langle \mathcal{A}u - \mathcal{A}v, w \rangle \leq L \|\nabla(u-v)\| \|\nabla w\|$

\implies existence and uniqueness of the weak solution $u^* \in H_0^1(\Omega)$

Energy framework

- Energy $\mathcal{E}(v) := \frac{1}{2} \int_{\Omega} \int_0^{|\nabla v(x)|^2} \mu(t) dt dx - F(v)$

- Distance $\mathbb{D}^2(u, v) := \mathcal{E}(v) - \mathcal{E}(u)$ **remark:** $\mathbb{D}^2(u^*, v) \approx \|\nabla(u^* - v)\|^2$

Adaptive FEM with nested iterative solvers

input: initial mesh \mathcal{T}_0 , initial guess $u_0^{0,0}$, adaptivity parameters $0 < \theta \leq 1$, $\lambda_{\text{lin}} > 0$

for each $\ell = 0, 1, 2, \dots$ **repeat** (mesh-refinement loop)

SOLVE & ESTIMATE (linearization loop)

for $k = 1, 2, \dots, K$, **repeat**

for $j = 1, 2, \dots, J$, **repeat** (algebra loop)

compute $u_\ell^{k,j} \approx u_\ell^{k,*}$ from the previous step $u_\ell^{k,j-1}$

compute the local indicators $\eta_\ell(T, u_\ell^k)$ for all $T \in \mathcal{T}_\ell$

until **[algebra-criterion]**

until $\mathbb{D}^2(u_\ell^{k,J}, u_\ell^{k-1,J}) \leq \lambda_{\text{lin}} \eta_\ell(u_\ell^{k,J})^2$

MARK choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^{K,J})^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^{K,J})^2$

REFINE $\mathcal{T}_{\ell+1} := \text{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

$u_{\ell+1}^{0,0} := u_\ell^{K,J}$

output: discrete solutions $u_\ell^{K,J}$ and corresponding estimators $\eta_\ell(u_\ell^{K,J})$

Termination of algebraic solver

- the stopping criterion should guarantee that nested linearization-algebraic solver **contracts in energy**

$$\mathbb{D}^2(u_\ell^*, u_\ell^{k,J}) \leq q_{\text{ctr}} \mathbb{D}^2(u_\ell^*, u_\ell^{k-1,J}) \quad 0 < q_{\text{ctr}} < 1$$

Equilibration criterion [HPSV21]

- stop algebraic solver if $\|\nabla(u_\ell^{k,J} - u_\ell^{k,J-1})\|^2 \leq \lambda_{\text{alg}} [\eta_\ell(u_\ell^{k,J})^2 + \mathbb{D}^2(u_\ell^{k,J}, u_\ell^{k,J-1})]$

- stop linearization if $\mathbb{D}^2(u_\ell^{K,J}, u_\ell^{K-1,J}) \leq \lambda_{\text{lin}} \eta_\ell(u_\ell^{K,J})$

\Rightarrow full R-linear convergence for arbitrary λ_{lin} but **sufficiently small** λ_{alg}

Note: there exists $C_{\text{nrg}}^* > 0$ st. $C_{\text{nrg}}^* \|\nabla(u_\ell^{k,*} - u_\ell^{k-1,J})\|^2 \leq \mathbb{D}^2(u_\ell^{k,*}, u_\ell^{k-1,J})$

Energy-based criterion [MPS24+]

- enforce algorithmically $\|\nabla(u_\ell^{k,J} - u_\ell^{k,J-1})\|^2 \lesssim \mathbb{D}^2(u_\ell^{k,J}, u_\ell^{k,J-1})$ (parameter-free)

- stop linearization if $\mathbb{D}^2(u_\ell^{K,J}, u_\ell^{K-1,J}) \leq \lambda_{\text{lin}} \eta_\ell(u_\ell^{K,J})$

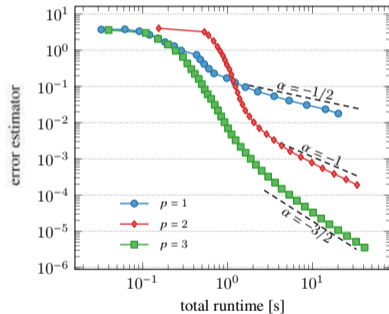
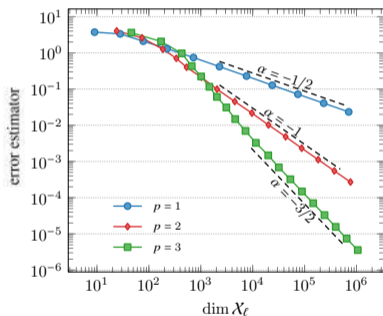
\Rightarrow full R-linear convergence for arbitrary $\lambda_{\text{lin}} > 0$

 Haberl, Praetorius, Schimanko, Vohralík: *Numer. Math.*, 147 (2021)

 Miraçi, Praetorius, Streitberger: *Math. Comp.* accepted (2025)

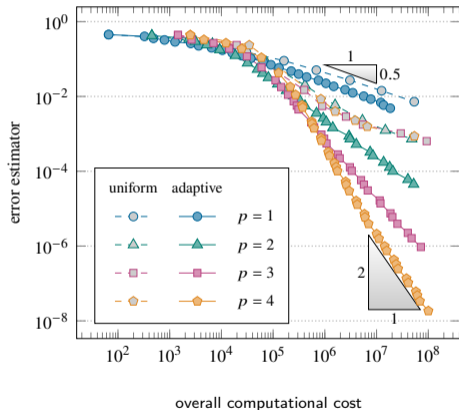
Optimality of AFEM with linearization : $p \geq 1$

$$-\operatorname{div}(\nabla u^* + \exp(-|\nabla u^*|^2)\nabla u^*) = 1 \quad \text{on } \Omega, \quad u^* = 0 \quad \text{in } \partial\Omega$$



Optimality of AFEM with symmetrization : $p \geq 1$

$$-\Delta u^*(x) + x \cdot \nabla u^*(x) + u^*(x) = 1 \quad \text{in } \Omega, \quad u^*(x) = 0 \quad \text{on } \partial\Omega$$



Take home messages

- 1 analysis of AFEM should focus rather on rates wrt. *complexity/time* than dofs
- 2 *linear complexity* and *contraction* of the iterative solver is crucial
- 3 use *nested* iterations and *termination* of solver balancing the different error components
- 4 *reliability* via a posteriori error estimators is ensured
- 5 full R-linear convergence
 - ▶ gives *contraction* regardless of algorithmic step
 - ▶ holds for *arbitrary* adaptivity parameters
 - ▶ provides the equivalence *rates = complexity*
- 6 *optimal complexity* is ensured for sufficiently small parameters
- 7 every *achievable* convergence rate is produced by AFEM
- 8 setting of certain non-symmetric and non-linear problems fits within this adaptive framework

Selected contributions

 Miraçi, Papež, Vohralík

A-posteriori-steered p -robust multigrid with optimal step-sizes and adaptive number of smoothing steps
SIAM J. Sci. Comput., 43, DOI: 10.1137/20M1349503 (2021)

 Innerberger, Miraçi, Praetorius, Streitberger

hp -robust multigrid solver on locally refined meshes for FEM discretizations of symmetric elliptic PDEs
ESAIM Math. Model. Numer. Anal., 58, DOI: 10.1051/m2an/2023104 (2024)

 Miraçi, Praetorius, Streitberger

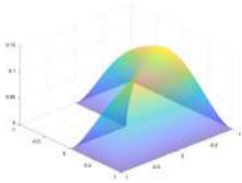
Unconditional full linear convergence and optimal complexity of adaptive iteratively linearized FEM for nonlinear PDEs
Math. Comp. accepted (2025)

 Bringmann, Miraçi, Praetorius

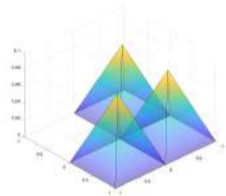
Chapter Four - Iterative solvers in adaptive FEM: Adaptivity yields quasi-optimal computational runtime

Advances in Applied Mechanics, Elsevier, 59, DOI:
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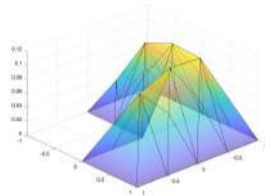
Thank you for your attention!



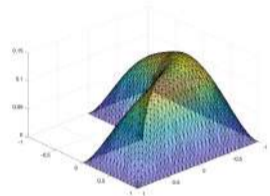
solution u^*



$\ell = 0$
 $k = 1$



$\ell = 1$
 $k = 3$



$\ell = 8$
 $k = 0$

