# AFEM with optimally preconditioned GMRES

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link to slides

# Workflow

MODELING

PDE

$u^\star \in \mathcal{X}$

estimator $\eta_\ell$ $\approx$ discretization error $\|u^\star - u_\ell^\star\|$ $\longrightarrow$

DISCRETIZATION($\ell$)

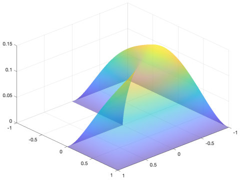
FEM (mesh $\mathcal{T}_\ell$, polynomial degree $p$)

$u_\ell^\star \in \mathcal{X}_\ell^p$

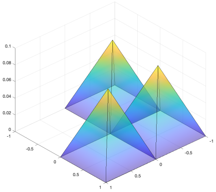
estimator $\zeta_\ell^k$ $\approx$ solver error $\|u_\ell^\star - u_\ell^k\|$ $\longrightarrow$

SOLVER($k$)

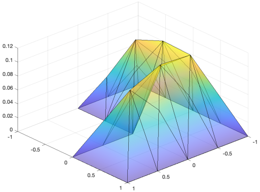
$u_\ell^k \in \mathcal{X}_\ell^p$



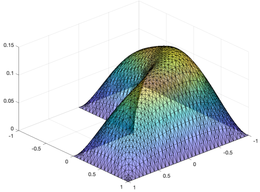
solution $u^\star$



$\ell = 0$
$k = 1$



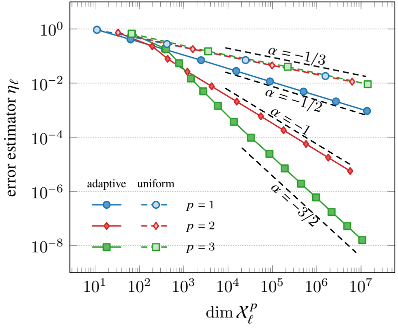
$\ell = 1$
$k = 3$

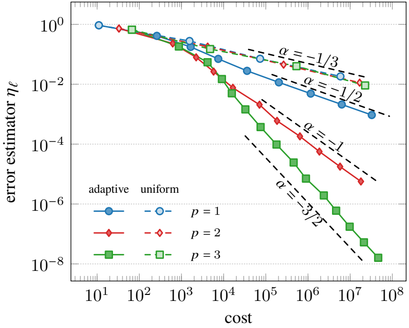


$\ell = 8$
$k = 0$

1. **Reliability** $\|u^\star - u_\ell^k\| \lesssim \eta_\ell + \zeta_\ell^k$
2. **Unconditional convergence** $u_\ell^k \longrightarrow u^\star$
3. **Optimal** rates of convergence

# Optimal convergence $\alpha = p/d$ wrt dofs and cost




$$\sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^{\alpha}\, \eta_\ell < \infty$$

$$\sup_{\ell \in \mathbb{N}_0} \Big( \sum_{\ell'=0}^{\ell} \sum_{k'=0}^{k[\ell']} \#\mathcal{T}_{\ell'} \Big)^{\alpha} \eta_\ell < \infty$$

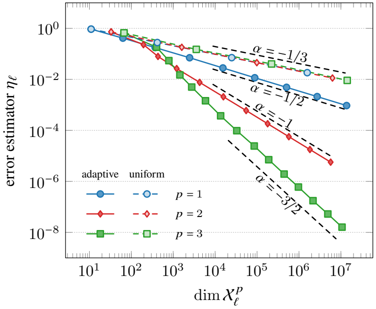

Binev, Dahmen, DeVore: *Numer. Math.*, 97 (2004)

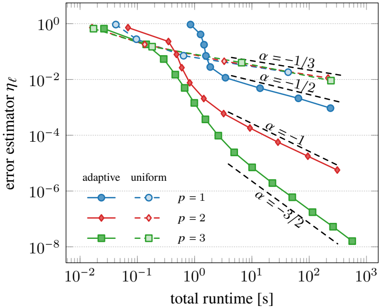
Stevenson: *Found. Comput. Math.*, 7 (2007)

Cascón, Kreuzer, Nochetto, Siebert: *SIAM J. Numer. Anal.*, 46 (2008)

Gantner, Haberl, Praetorius, Schimanko: *Math. Comp.*, 90 (2021)

# Towards optimal complexity




*Optimal complexity of AFEM requires each of its modules to be realized in linear complexity:*

- SOLVE is critical
- ESTIMATE ✓
- MARK (Stevenson 2007, Pfeiler-Praetorius 2020 for minimal cardinality marking) ✓
- REFINE (Binev-Dahmen-DeVore 2004, Stevenson 2008) ✓

# Optimal algebraic solver

# Model problem

Consider the **non-symmetric linear elliptic PDE**

$$-\operatorname{div}(\boldsymbol{K}\nabla u^\star) + \boldsymbol{b}\cdot\nabla u^\star + cu^\star = f - \operatorname{div}\boldsymbol{f} \quad \text{in } \Omega \quad \text{with } u^\star = 0 \quad \text{on } \partial\Omega$$

**Weak formulation:** search for $u^\star \in \mathcal{X} := H^1_0(\Omega)$ solution of

$$b(u^\star, v) = \langle f, v\rangle_{L^2(\Omega)} + \langle \boldsymbol{f}, \nabla v\rangle_{L^2(\Omega)} =: F(v) \quad \text{for all } v \in \mathcal{X}$$

**Notation:**

$$a(v,w) := \langle \boldsymbol{K}\,\nabla v, \nabla w\rangle_{L^2(\Omega)} \qquad |\!|\!| v |\!|\!| := a(v,v)^{1/2}$$

$$b(v,w) := a(v,w) + \langle \boldsymbol{b}\cdot\nabla v, w\rangle_{L^2(\Omega)} + \langle c\,v, w\rangle_{L^2(\Omega)} \quad \text{for all } v, w \in \mathcal{X}$$

*Lax-Milgram framework* $\Longrightarrow$ *well-posed problem*

$$|b(v,w)| \le C_{\mathsf{bnd}}\, |\!|\!| v |\!|\!|\, |\!|\!| w |\!|\!| \quad \text{and} \quad b(v,v) \ge C_{\mathsf{ell}}\, |\!|\!| v |\!|\!|^2 \quad \text{for all } v, w \in \mathcal{X}$$

# FEM discretization

### Functional writing

- $\mathcal{T}_\ell$ simplicial mesh of $\Omega$ and $p \geq 1$ polynomial degree
- finite element space $\mathcal{X}_\ell^p := \{v_\ell \in H_0^1(\Omega) : v_\ell|_T \in \mathbb{P}_p(T)\ \forall T \in \mathcal{T}_\ell\} \subset \mathcal{X}$
- search for $u_\ell^\star \in \mathcal{X}_\ell^p$ solution of

$$b(u_\ell^\star, v_\ell) = F(v_\ell) \quad \forall v_\ell \in \mathcal{X}_\ell^p$$

### Discrete writing:

- $N_\ell := \dim \mathcal{X}_\ell^p$
- $\varphi_{\ell,1}, \dots, \varphi_{\ell,N_\ell}$ basis of $\mathcal{X}_\ell^p$
- $(\boldsymbol{B}_\ell)_{j,k} := b(\varphi_{\ell,k}, \varphi_{\ell,j}) \quad (\boldsymbol{d}_\ell)_k := F(\varphi_{\ell,k}) \quad (\boldsymbol{A}_\ell)_{j,k} := a(\varphi_{\ell,k}, \varphi_{\ell,j})$
- search for $\boldsymbol{x}_\ell^\star \in \mathbb{R}^{N_\ell}$ solution of

$$\boldsymbol{B}_\ell\, \boldsymbol{x}_\ell^\star = \boldsymbol{d}_\ell$$

### Link:

$$u_\ell^\star := \sum_{j=1}^{N_\ell} (\boldsymbol{x}_\ell^\star)_j \varphi_{\ell,j}$$

# Optimal additive Schwarz

**Preconditioner requirements:**

- symmetric and positive definite preconditioner $\boldsymbol{P}_\ell \approx \boldsymbol{B}_\ell^{-1}$

$$\|\boldsymbol{x}_\ell\|_{\boldsymbol{P}_\ell^{-1}} := (\boldsymbol{x}_\ell, \boldsymbol{x}_\ell)_{\boldsymbol{P}_\ell^{-1}}^{1/2} := (\boldsymbol{P}_\ell^{-1}\boldsymbol{x}_\ell, \boldsymbol{x}_\ell)_2^{1/2}$$

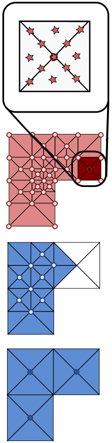
- linear complexity of applying $\boldsymbol{P}_\ell$
- discrete-functional equivalence

$$C_b^{-1}\,\|v_\ell\| \le \|\boldsymbol{x}_\ell\|_{\boldsymbol{P}_\ell^{-1}} \le C_b\,\|v_\ell\| \quad \text{for all } v_\ell = \sum_{j=1}^{N_\ell} (\boldsymbol{x}_\ell)_j\,\varphi_{\ell,j} \in \mathcal{X}_\ell^p$$

**Subspace correction via additive Schwarz:**

$h$-robustness | $p$-robustness

$$\boldsymbol{P}_\ell^{AS} := \underbrace{\boldsymbol{I}_0^1\,(\boldsymbol{A}_0^1)^{-1}}_{\text{lowest-order global solve}}(\boldsymbol{I}_0^1)^T + \sum_{\ell'=1}^{\ell-1}\boldsymbol{I}_{\ell'}^1\,\underbrace{(\boldsymbol{D}_{\ell'}^{+})^{-1}}_{\text{lowest-order local solves}}(\boldsymbol{I}_{\ell'}^1)^T + \sum_{z\in\mathcal{V}_\ell}\boldsymbol{I}_{\ell,z}^p\,\underbrace{(\boldsymbol{A}_{\ell,z}^p)^{-1}}_{\text{high-order patch-wise solves}}\,(\boldsymbol{I}_{\ell,z}^p)^T$$




Chen, Nochetto, Xu: *Numer. Math.*, 120 (2012)

Schöberl, Melenk, Pechstein, Zaglmayr: *IMA J. Numer. Anal.*, 28 (2008)

Hilbert, Miraçi, Praetorius: arXiv: 2601.14911 (2026)

# PGMRES with restart

**input:** system matrix $\boldsymbol{B}_\ell$, preconditioner $\boldsymbol{P}_\ell$, right-hand side $\boldsymbol{d}_\ell$, initial guess $\boldsymbol{x}_\ell^0$, $\Lambda : \mathbb{R}^{N_\ell} \to \mathbb{R}_{\geq 0}$, $k_{\max} \geq 1$

initialize $\boldsymbol{r}_\ell^0 := \boldsymbol{d}_\ell - \boldsymbol{B}_\ell \boldsymbol{x}_\ell^0$ $\quad \boldsymbol{s}_\ell^0 := \boldsymbol{P}_\ell \boldsymbol{r}_\ell^0$ $\quad \widetilde{\boldsymbol{v}}^1 := \boldsymbol{r}_\ell^0 / \|\boldsymbol{s}_\ell^0\|_{\boldsymbol{P}_\ell^{-1}} = \boldsymbol{r}_\ell^0 / \langle \boldsymbol{s}_\ell^0, \boldsymbol{r}_\ell^0 \rangle_2^{1/2}$ $\quad \boldsymbol{v}^1 := \boldsymbol{P}_\ell \widetilde{\boldsymbol{v}}^1$ $\quad k := -1$

**for** each $\widetilde{k} = 0, 1, 2, \ldots, k_{\max}$ **repeat** (residual minimization loop)

$\widetilde{\boldsymbol{w}} := \boldsymbol{B}_\ell \boldsymbol{v}^{\widetilde{k}}$ $\quad \boldsymbol{w} := \boldsymbol{P}_\ell \widetilde{\boldsymbol{w}}$ $\quad k \leftarrow k+1$ $\quad R \leftarrow \lfloor k / k_{\max} \rfloor$

**for** $j = 1, 2, \ldots, \widetilde{k}$, **repeat** (Arnoldi process loop)

$\boldsymbol{H}_{j,\widetilde{k}} := \langle \boldsymbol{w}, \boldsymbol{v}^j \rangle_{\boldsymbol{P}_\ell^{-1}} = \langle \widetilde{\boldsymbol{w}}, \boldsymbol{v}^j \rangle_2$ $\quad \widetilde{\boldsymbol{w}} := \widetilde{\boldsymbol{w}} - \boldsymbol{H}_{j,\widetilde{k}} \widetilde{\boldsymbol{v}}^j$ $\quad \boldsymbol{w} := \boldsymbol{w} - \boldsymbol{H}_{j,\widetilde{k}} \boldsymbol{v}^j$

**end**

$\boldsymbol{H}_{\widetilde{k}+1,\widetilde{k}} := \|\boldsymbol{w}\|_{\boldsymbol{P}_\ell^{-1}} = \langle \widetilde{\boldsymbol{w}}, \boldsymbol{w} \rangle_2^{1/2}$ $\quad \boldsymbol{y} := \arg\min_{\boldsymbol{z} \in \mathbb{R}^{\widetilde{k}}} \left\| \|\boldsymbol{s}_\ell^{Rk_{\max}}\|_{\boldsymbol{P}_\ell^{-1}} \boldsymbol{e}^1 - \boldsymbol{H}\boldsymbol{z} \right\|_2$

$\boldsymbol{x}_\ell^k := \boldsymbol{x}_\ell^{Rk_{\max}} + \sum_{j=1}^{\widetilde{k}} \boldsymbol{y}_j \boldsymbol{v}^j$ $\quad \boldsymbol{r}_\ell^k := \boldsymbol{d}_\ell - \boldsymbol{B}_\ell \boldsymbol{x}_\ell^k$ $\quad \boldsymbol{s}_\ell^k := \boldsymbol{P}_\ell \boldsymbol{r}_\ell^k$ $\quad \|\boldsymbol{s}_\ell^k\|_{\boldsymbol{P}_\ell^{-1}} = \langle \boldsymbol{r}_\ell^k, \boldsymbol{s}_\ell^k \rangle_2^{1/2}$

**if** $\widetilde{k} \neq k_{\max}$ **then** $\widetilde{\boldsymbol{v}}^{\widetilde{k}+1} := \widetilde{\boldsymbol{w}} / \boldsymbol{H}_{\widetilde{k}+1,\widetilde{k}}$, $\boldsymbol{v}^{\widetilde{k}+1} := \boldsymbol{w} / \boldsymbol{H}_{\widetilde{k}+1,\widetilde{k}}$ **else** new basis from $\boldsymbol{r}_\ell^k$

**until** $\|\boldsymbol{s}_\ell^k\|_{\boldsymbol{P}_\ell^{-1}} \leq \Lambda(\boldsymbol{x}_\ell^k)$

set $\underline{k}[\ell] := k$

**output:** approximation $\boldsymbol{x}_\ell^{\underline{k}[\ell]}$, restart index $R$, preconditioned residual norm $\|\boldsymbol{s}_\ell^{\underline{k}[\ell]}\|_{\boldsymbol{P}_\ell^{-1}}$

Sarkis, Szyld: *Comput. Methods Appl. Mech. Engrg.*, 196 (2007)

# A posteriori estimator and solver contraction

**Proposition 1** ($h$- and $p$-robust reliability and efficiency of the a posteriori estimator)

$$C_{\text{ell}}\, C_b^{-2}\, \|u_\ell^\star - u_\ell^k\| \le \|\boldsymbol{s}_\ell^k\|_{\boldsymbol{P}_\ell^{-1}} =: \zeta_\ell^k \le C_{\text{bnd}}\, C_b^2\, \|u_\ell^\star - u_\ell^k\|$$

**Proposition 2** ($h$- and $p$-robust contraction)

$$\|\boldsymbol{s}_\ell^{k+1}\|_{\boldsymbol{P}_\ell^{-1}} \le q_{\text{alg}}\, \|\boldsymbol{s}_\ell^k\|_{\boldsymbol{P}_\ell^{-1}} \qquad 0 < q_{\text{alg}}(C_{\text{bnd}}, C_{\text{ell}}, C_b) < 1$$

**Note**: result holds for any $k_{\max} \ge 1$ $\Longrightarrow$ linear complexity

**Challenge**: contraction yes, but in preconditioner norm: $\Longrightarrow$ even if $\|v_{\ell+1}^0\| = \|v_\ell^k\|$

$\Longrightarrow$ typically $\|\boldsymbol{s}_{\ell+1}^0\|_{\boldsymbol{P}_{\ell+1}^{-1}} \neq \|\boldsymbol{s}_\ell^k\|_{\boldsymbol{P}_\ell^{-1}}$


- Eisenstat, Elman, Schultz: *SIAM J. Numer. Anal.*, 20 (1983)
- Sarkis, Szyld: *Comput. Methods Appl. Mech. Engrg.*, 196 (2007)
- Schöberl, Melenk, Pechstein, Zaglmayr: *IMA J. Numer. Anal.*, 28 (2008)
- Miraçi, Papež, Vohralík: *SIAM J. Sci. Comput.*, 43 (2021)
- Innerberger, Miraçi, Praetorius, Streitberger: *ESAIM Math. Model. Numer. Anal.*, 58 (2024)
- Führer, Hilbert, Miraçi, Praetorius: (in preparation)

# Adaptive mesh refinement and algebraic solver

# AFEM with iterative solver

**input:** initial mesh $\mathcal{T}_0$, initial guess $u_0^0$, adaptivity parameters $0 < \theta \leq 1$, $\lambda_{\mathrm{alg}}$, $C_{\mathrm{alg}}$, $\mathsf{S} := 0$

**for** each $\ell = 0, 1, 2, \ldots$ **repeat** (mesh-refinement loop)

**SOLVE & ESTIMATE** (algebraic solver loop)

compute $u_\ell^k$ and $\|\boldsymbol{s}_\ell^k\|_{\boldsymbol{P}_\ell^{-1}}$ from $u_\ell^0$ via PGMRES with $\Lambda(\boldsymbol{x}_\ell^k) := \lambda_{\mathrm{alg}}\, \eta_\ell(u_\ell^k)$ for $k \leq \underline{k}[\ell]$

**ADAPTIVE PARAMETER CONTROL**

- **if** $\mathsf{S} > C_{\mathrm{alg}}\,(\eta_\ell(u_\ell^k) + \|\boldsymbol{s}_\ell^k\|_{\boldsymbol{P}_\ell^{-1}})^{-1}$ **then** $C_{\mathrm{alg}} \leftarrow 2\, C_{\mathrm{alg}}$ $\lambda_{\mathrm{alg}} \leftarrow \lambda_{\mathrm{alg}}/2$
- $\mathsf{S} \leftarrow \mathsf{S} + (\eta_\ell(u_\ell^k) + \|\boldsymbol{s}_\ell^k\|_{\boldsymbol{P}_\ell^{-1}})^{-1}$

**MARK** select quasi-minimal $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T, u_\ell^k)^2 \leq \sum_{T\in\mathcal{M}_\ell} \eta_\ell(T, u_\ell^k)^2$

**REFINE** $\mathcal{T}_{\ell+1} := \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

$u_{\ell+1}^0 := u_\ell^k$

**output:** approximations $u_\ell^k$ and corresponding estimators $\zeta_\ell^k = \|\boldsymbol{s}_\ell^k\|_{\boldsymbol{P}_\ell^{-1}}, \eta_\ell(u_\ell^k)$

# Reliability and linear convergence

A posteriori control for each approximation $u_\ell^k$

$$\underbrace{|\!|\!|(u^\star - u_\ell^k)|\!|\!|}_{\text{total error}} \le \underbrace{|\!|\!|(u^\star - u_\ell^\star)|\!|\!|}_{\text{discretization error}} + \underbrace{|\!|\!|(u_\ell^\star - u_\ell^k)|\!|\!|}_{\text{algebraic error}} \overset{\text{reliability (discretization)}}{\lesssim} \underbrace{\eta_\ell(u_\ell^\star)}_{\text{discretization estimator}} + |\!|\!|(u_\ell^\star - u_\ell^k)|\!|\!|$$

$$\overset{\text{reliability (algebra)}}{\lesssim} \eta_\ell(u_\ell^k) + \underbrace{\|\boldsymbol{s}_\ell^k\|_{\boldsymbol{P}_\ell^{-1}}}_{\text{algebra estimator}} =: \underbrace{\mathsf{H}_\ell^k}_{\text{quasi-error}}$$

Define the index set $\mathcal{Q} := \{(\ell, k) \in \mathbb{N}^2 :\ u_\ell^k \text{ is computed by algorithm}\}$
with ordering $|\ell, k| := \#\{(\ell', k') \in \mathcal{Q} \colon u_{\ell'}^{k'} \text{ computed earlier than } u_\ell^k\}$

**Definition** (full R-linear convergence of the quasi-error)

$$\mathsf{H}_\ell^k \le C_{\text{lin}}\, q_{\text{lin}}^{|\ell,k| - |\ell',k'|}\, \mathsf{H}_{\ell'}^{k'}, \quad C_{\text{lin}} > 0,\ 0 < q_{\text{lin}} < 1$$

*Contraction independently of the algorithmic step: mesh-refinement or algebraic iteration*

**Corollary** (convergence of the total error)

$$|\!|\!|u^\star - u_\ell^k|\!|\!| \lesssim \mathsf{H}_\ell^k \lesssim q_{\text{lin}}^{|\ell,k|}\, \mathsf{H}_0^0 \longrightarrow 0 \quad \text{for} \quad |\ell, k| \longrightarrow \infty$$

# Conditional full R-linear convergence

**Lemma** (summability criteria and $R$-linear convergence) Let $a_\ell \in \mathbb{R}_{\geq 0}$, $\ell \in \mathbb{N}_0$, $\beta, \gamma > 0$. Then (i)–(iii) are pairwise equivalent:

**(i)** *$R$-linear convergence:* $a_\ell \leq C_{\mathrm{lin}}\, q_{\mathrm{lin}}^{\ell-\ell'}\, a_{\ell'}$ for all $0 \leq \ell' \leq \ell$ $\quad 0 < q_{\mathrm{lin}} < 1, C_{\mathrm{lin}} > 1$

**(ii)** *tail summability:* $\displaystyle\sum_{\ell'=\ell+1}^{\infty} a_{\ell'}^{\beta} \leq C_\beta\, a_\ell^{\beta}$ for all $\ell \in \mathbb{N}_0$ $\quad C_\beta > 0$

**(iii)** *inverse summability:* $\displaystyle\sum_{\ell'=0}^{\ell-1} a_{\ell'}^{-\gamma} \leq C_\gamma\, a_\ell^{-\gamma}$ for all $\ell \in \mathbb{N}_0$ $\quad C_\gamma > 0$

If (i)–(iii) hold for $\ell \geq \ell_0 \in \mathbb{N}$, then they hold for all $\ell \in \mathbb{N}_0$

**Proposition** (conditional tail summability in the levels) Let $\lambda^\star := \min\{1, C_{\mathsf{stab}}^{-1} C_{\mathsf{ell}}\, C_b^{-2}\}$. If there exists $\ell_0 \in \mathbb{N}$ such that $\quad 0 < \lambda_{\mathsf{alg}} < \lambda^\star \theta^{1/2}$ $\quad$ for all $\ell \geq \ell_0$, then

$$\sum_{\ell'=\ell+1}^{\infty} \mathsf{H}_{\ell'}^{\underline{k}} \leq C_{\mathsf{tail}}\, \mathsf{H}_{\ell}^{\underline{k}} \qquad C_{\mathsf{tail}} \geq 1$$

Carstensen, Feischl, Page, Praetorius: *Comput. Math. Appl.*, 67 (2014)

# Unconditional full R-linear convergence

**ADAPTIVE PARAMETER CONTROL:** $\mathsf{H}_\ell^k = \eta_\ell(u_\ell^k) + \|\boldsymbol{s}_\ell^k\|_{\boldsymbol{P}_\ell^{-1}}$ *computable*

- **if** $\mathsf{S} = \sum_{\ell'=0}^{\ell-1} (\mathsf{H}_{\ell'}^k)^{-1} > C_{\mathsf{alg}} (\mathsf{H}_\ell^k)^{-1}$ **then** $C_{\mathsf{alg}} \leftarrow 2\, C_{\mathsf{alg}}$ $\lambda_{\mathsf{alg}} \leftarrow \lambda_{\mathsf{alg}}/2$
- $\mathsf{S} \leftarrow \mathsf{S} + (\mathsf{H}_\ell^k)^{-1}$

$\Longrightarrow$ *new a-posteriori-steered criterion algorithmically enforces the inverse summability criterion*

**Lemma** (finitely many updates) The input parameters $C_{\mathrm{alg}}$ and $\lambda_{\mathrm{alg}}$ are updated only finitely many times.

**Theorem** (unconditional full R-linear convergence of the quasi-error) Consider arbitrary $0 < \theta \leq 1$, $\lambda > 0$. Then

$$\mathsf{H}_\ell^k \leq C_{\mathrm{lin}}\, q_{\mathrm{lin}}^{|\ell,k|-|\ell',k'|}\, \mathsf{H}_{\ell'}^{k'}, \quad C_{\mathrm{lin}} > 0, \; 0 < q_{\mathrm{lin}} < 1$$

Führer, Hilbert, Miraçi, Praetorius: (in preparation)

# Full R-linear convergence implies rates = complexity

- $\mathfrak{R}(\alpha) := \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^\alpha \mathsf{H}_\ell^k < \infty$ rate $\alpha$ wrt dofs is possible
- $\widehat{\mathfrak{R}}(\alpha) := \sup_{(\ell,k)\in\mathcal{Q}} \Big( \sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^\alpha \mathsf{H}_\ell^k < \infty$ rate $\alpha$ wrt costs/overall computational cost is possible

**Proposition**

$$\mathfrak{R}(\alpha) \le \widehat{\mathfrak{R}}(\alpha) \le \frac{C_{\mathrm{lin}}}{(1-q_{\mathrm{lin}}^{1/\alpha})^\alpha} \mathfrak{R}(\alpha)$$

- Proof: $\#\mathcal{T}_{\ell'} \le \mathfrak{R}(\alpha)^{\frac{1}{\alpha}} (\mathsf{H}_{\ell'}^{k'})^{-\frac{1}{\alpha}} \quad \forall\, (\ell',k') \in \mathcal{Q}$, summing and using the geometric series :

$$\Longrightarrow \sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \le \mathfrak{R}(\alpha)^{\frac{1}{\alpha}} \sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} (\mathsf{H}_{\ell'}^{k'})^{-\frac{1}{\alpha}} \le \mathfrak{R}(\alpha)^{\frac{1}{\alpha}} C_{\mathrm{lin}}^{\frac{1}{\alpha}} \Big( \sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} q_{\mathrm{lin}}{}^{(|\ell,k|-|\ell',k'|)/\alpha} \Big) (\mathsf{H}_\ell^k)^{-\frac{1}{\alpha}}$$

Carstensen, Feischl, Page, Praetorius: *Comput. Math. Appl.*, 67 (2014)

# Optimal complexity

We introduce the notion of *approximation class* : $\|u^\star\|_{\mathbb{A}_\alpha} := \sup_{N \geq \#\mathcal{T}_0} N^\alpha [\min_{\#\mathcal{T}_{\rm opt} \leq N} \eta_{\rm opt}]$

$$\|u^\star\|_{\mathbb{A}_\alpha} < \infty \iff u^\star \text{ can be approximated with rate } \alpha \text{ wrt dofs}$$

**Theorem** (optimal convergence wrt to overall computational cost)
Let $\alpha > 0$ such that $\|u^\star\|_{\mathbb{A}_\alpha} < \infty$. Suppose $0 < \theta < 1$ and $\lambda_{\rm alg} > 0$ sufficiently small

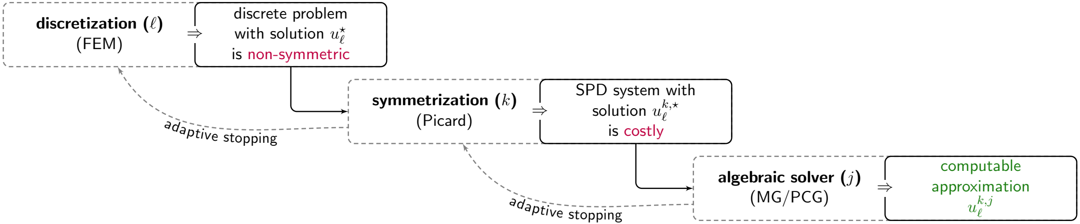
$$\Longrightarrow \quad \|u^\star\|_{\mathbb{A}_\alpha} \lesssim \sup_{(\ell,k)\in\mathcal{Q}} \Big( \sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\leq|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^\alpha \mathsf{H}_\ell^k \lesssim \max\{\|u^\star\|_{\mathbb{A}_\alpha}, \mathsf{H}_0^0)\}$$

*if* $u^\star$ *can be approximated with rate* $\alpha$ *wrt dofs*
*then AFEM with optimal algebraic solver approximates* $u^\star$ *with the rate* $\alpha$ *wrt cost.*


Gantner, Haberl, Praetorius, Schimanko: *Math. Comp.*, 90 (2021)

Führer, Hilbert, Miraçi, Praetorius: (in preparation)

# Adaptive approach with nested solvers




$\Longrightarrow$ three nested loops: mesh-refinement ($\ell$) $\mapsto$ symmetrization ($k$) $\mapsto$ algebra ($j$)

$\Longrightarrow$ each of the solvers (algebra and symmetrization) is contractive


Congreve, Wihler: *J. Comput. Appl. Math.*, 311 (2017)

Heid, Praetorius, Wihler: *Comput. Methods Appl. Math.*, 21 (2021)

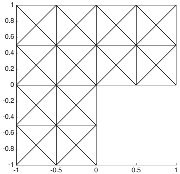
Haberl, Praetorius, Schimanko, Vohralík: *Numer. Math.*, 147 (2021)

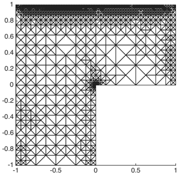
Brunner, Innerberger, Miraçi, Praetorius, Streitberger, Heid: *IMA J. Numer. Anal.*, 44 (2024)

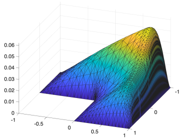
# Numerical experiments

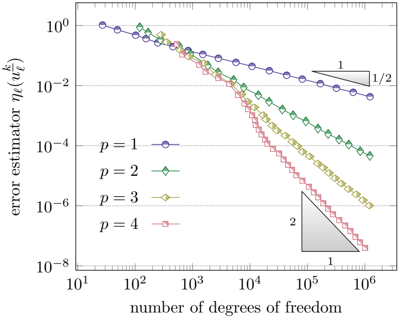
# Rate-optimality of AFEM

$$-\Delta u^\star + \begin{pmatrix} 1 \\ 25 \end{pmatrix} \cdot \nabla u^\star = 1 \quad \text{in} \quad \Omega.$$

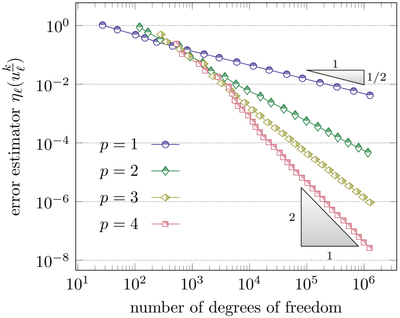






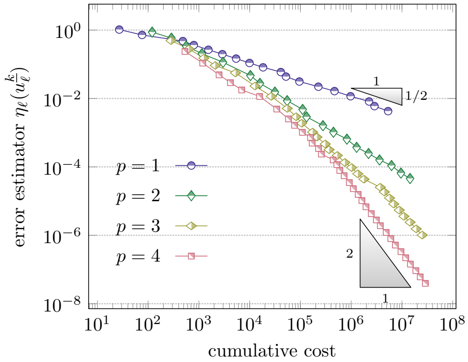



$k_{\max} = 5$

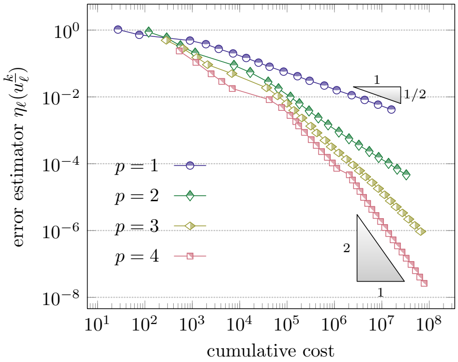



$k_{\max} = 1$

# Cost-optimality of AFEM

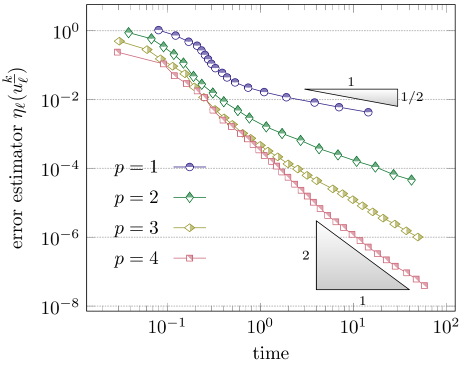



$k_{\max} = 5$

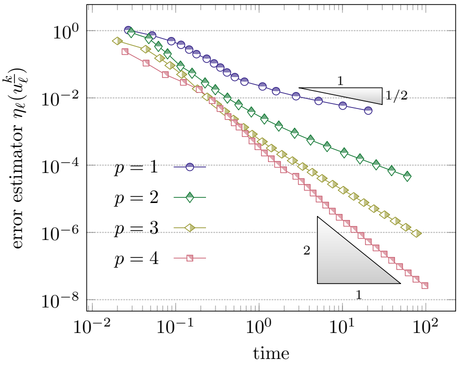



$k_{\max} = 1$

# Optimal complexity of AFEM

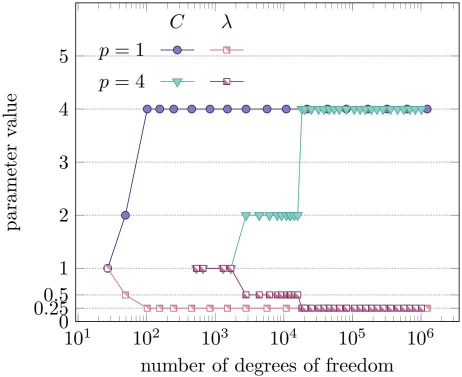



$k_{\max} = 5$

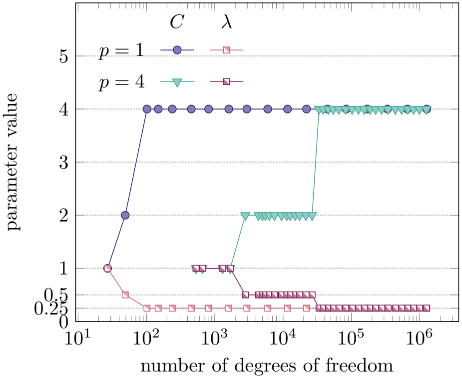



$k_{\max} = 1$

# Parameter control

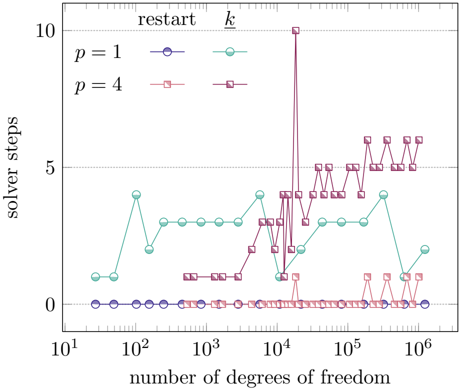



$k_{\max} = 5$

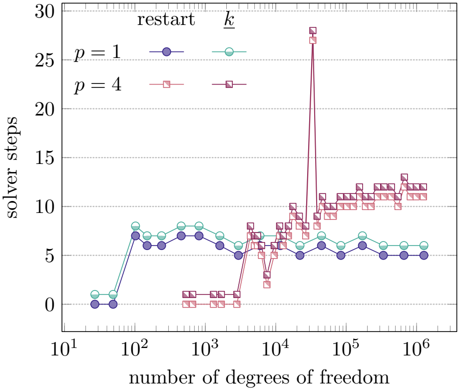



$k_{\max} = 1$

# PGMRES steps



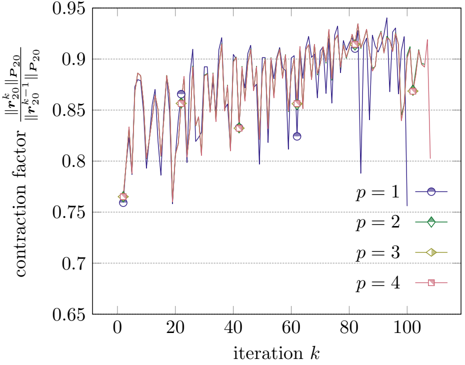

$k_{\max} = 5$



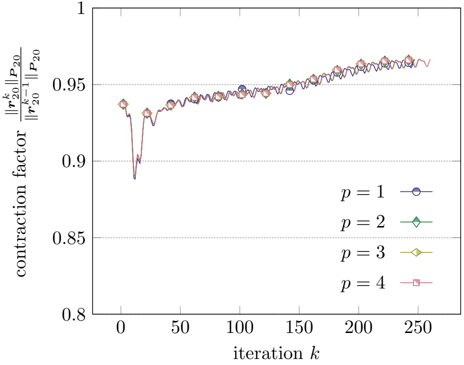

$k_{\max} = 1$

# PGMRES robustness



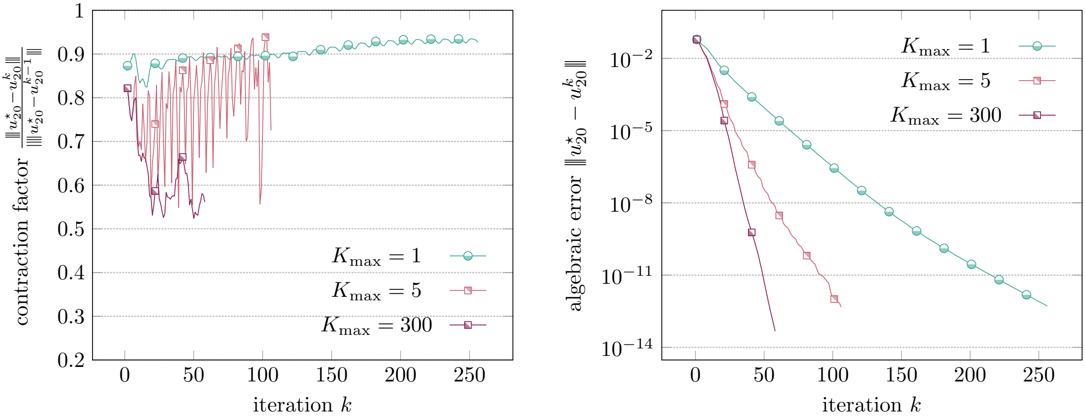



$k_{\max} = 5$




$k_{\max} = 1$

# PGMRES restart and energy error study



# Take home messages

SCIENCES SORBONNE UNIVERSITÉ

1. analysis of AFEM for non-symmetric problems *without nested solvers*
2. *linear complexity* and *optimality* of the preconditioner is crucial
3. PGMRES *contracts* in preconditioned norm even with *restart* after each step
4. use *nested* iterations and *termination* of solver balancing the different error components
5. *reliability* via a posteriori error estimators is ensured at each algorithmic step
6. *new adaptive control* on parameters enforcing algorithmically full R-linear convergence
   - gives *contraction* regardless of algorithmic step
   - holds for *arbitrary* adaptivity parameters
   - provides the equivalence *rates* = *complexity*
7. *optimal complexity* is ensured for sufficiently small parameters

# Selected contributions

SCIENCES SORBONNE UNIVERSITÉ


Miraçi, Papež, Vohralík

A-posteriori-steered $p$-robust multigrid with optimal step-sizes and adaptive number of smoothing steps
*SIAM J. Sci. Comput.*, 43, DOI: 10.1137/20M1349503 (2021)

Innerberger, Miraçi, Praetorius, Streitberger

$hp$-robust multigrid solver on locally refined meshes for FEM discretizations of symmetric elliptic PDEs
*ESAIM Math. Model. Numer. Anal.*, 58, DOI: 10.1051/m2an/2023104 (2024)

Bringmann, Miraçi, Praetorius

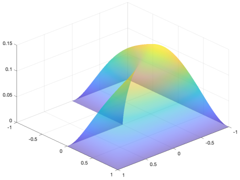
Chapter Four - Iterative solvers in adaptive FEM: Adaptivity yields quasi-optimal computational runtime
*Advances in Applied Mechanics, Elsevier*, 59, DOI: https://doi.org/10.1016/bs.aams.2024.08.002 (2024)

Führer, Hilbert, Miraçi, Praetorius

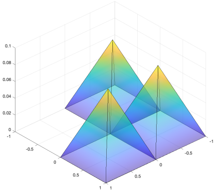
AFEM with optimally preconditioned GMRES yields optimal complexity
(in preparation)

# Thank you for your attention!

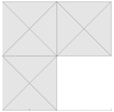


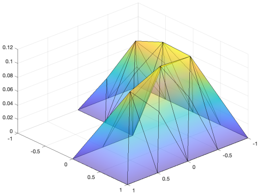


solution $u^\star$

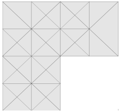


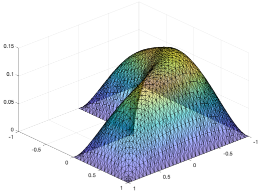
$\ell = 0$
$k = 1$





$\ell = 1$
$k = 3$





$\ell = 8$
$k = 0$